THEMATICS
4 1940

Mathematical Reviews

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Published monthly by

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Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 1, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879.

Mathematical Reviews

Vol. 1, No. 10 OCTOBER, 1940 Pages 289-320

HISTORY

Wygodski, M. Mathematics of the ancient Babylonians. Uspekhi Matem. Nauk **7**, 102–153 (1940). (Russian) [MF 2274]

Chatley, Herbert. The Egyptian celestial diagram. Observatory **63**, 68–72 (1940). [MF 2313]
The later list of the 36 decans is given.
O. Neugebauer (Providence, R. I.).

Singh, A. N. Hindu trigonometry. Proc. Benares Math. Soc. **1**, 77–92 (1939). [MF 1534]
The author gives an outline of Hindu trigonometry as it is found in the astronomical works ranging from Âryabhaṭa (499) to Kamalâkara (1685). He starts by giving an etymology of the modern conceptions of sine, cosine and versed sine, and then transcribes trigonometrical theorems contained in the different Siddhântas into modern notation. In the paper there is no discrimination between the two Sûrya-Siddhântas, namely the one (about 400) whose content is partly known through the Pañcasiddhântikâ and the one which is still extant and which is centuries later. The question of a possible dependence between the Greek and the Hindu trigonometry is not discussed. *O. Schmidt.*

Fujiwara, M. Miscellaneous notes on the history of Chinese mathematics. I. Tôhoku Math. J. **46**, 284–294 (1940). (Japanese) [MF 2442]

Fujiwara, M. Miscellaneous notes on the history of Chinese mathematics. II. Tôhoku Math. J. **47**, 35–48 (1940). (Japanese) [MF 2621]

Fujiwara, M. Miscellaneous notes on the history of Wazan. III. Tôhoku Math. J. **46**, 295–308 (1940). (Japanese) [MF 2443]

Fujiwara, M. Miscellaneous notes on the history of Wazan. IV. Tôhoku Math. J. **47**, 49–57 (1940). (Japanese) [MF 2622]

Minoda, Takashi. On "Katuyô Sampô, Book III" of Takakazu Seki. Tôhoku Math. J. **47**, 99–109 (1940). (Japanese) [MF 2629]

***Rey, Abel. La Maturité de la Pensée Scientifique en Grèce.** L'Évolution de l'Humanité, Synthèse Collective. Première Section, Série Complémentaire: La Science dans l'Antiquité, vol. 3. Editions Albin Michel, Paris, 1939. xxi+574 pp. 45 fr.
This book is the third of a series of four volumes devoted to the history of science in antiquity. Volume one, concerning oriental science, appeared in 1930 and volume two, on the earliest Greek period, in 1939. Naturally the history of mathematics and astronomy plays an important role in this work, although the present volume is mainly concerned with Greek philosophy. The text is written in an excellent fluent style and gives a very consistent picture of the intellectual history of the classical period of Greek culture. In the opinion of the reviewer, this consistency is mainly due to the contradictory, unreliable and extremely fragmentary character of our sources. As far as the mathematics is concerned, it may be the fault of the reviewer, when he feels himself unable to understand sentences as the following: "La cohérence de la conception géométrique (les dimensions réelles des points, longueurs, etc.) va donc augmentant aux dépens de la conception héritée de l'Ionisme. Les principes réalistes se vident au profit des principes arithmo-géométriques." But in the comparatively few places where concrete facts are mentioned almost everything is wrong: formulae misprinted or senseless [for example, p. 367, ff.], well-known historical facts misinterpreted or neglected (for instance, there appears, of course, the old story of the ruler and compass [p. 164, ff.], and it is declared that only "peu de chose" concerning Babylonian calculations is known [p. 365]), or such commentaries are given as, for example, on page 358, where the use of place value notation is explained as "l'utilisation à peu près intuitive de la notion de relativité."

At the end eight pages of bibliography are given, rather carelessly as regards exactness and arrangement of quotations. Even Tannery's works are incorrectly quoted. In spite of the fact that only "ouvrages fondamentaux" are claimed to be quoted, the main part of the bibliography contains papers which are absolutely out of date. On the other hand, very important books, such as W. Jäger's "Aristoteles," are missing. *O. Neugebauer* (Providence, R. I.).

van der Waerden, B. L. Zenon und die Grundlagenkrise der griechischen Mathematik. Math. Ann. **117**, 141–161 (1940). [MF 2151]
In the first paragraph the author shows that the famous paradoxa of Zeno (for example, of the tortoise and Achilles) are not at all directed against the infinite divisibility of geometrical magnitudes, but that their aim is simply to support the assumption of Parmenides that all movement is only a human fiction. The second part points out that in Zeno's time no mathematical theory of importance existed in which infinitesimal methods played a role. This fits in with the general concept of the development of Greek mathematics, which is familiar, at least since E. Frank's book "Plato und die sogenannten Pythagoreer" [Halle, 1923]. The last paragraph emphasizes that the so-called "crisis" of the foundation of Greek mathematics did not originate in the problem of infinite divisibility but from the discovery of irrationals. *O. Neugebauer.*

Conte, Luigi. Dalla "sezione del cono" di Sereno. III. Period. Mat. (4) **20**, 1–23 (1940). [MF 1724]
In the same journal [**18**, 70–86 (1938)], the author started to publish a translation of and commentary on the "conic

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sections" of Serenus [as far as the reviewer can see, without knowledge of the recent extensively commented translation by Ver Eecke, Bruxelles, 1929]. The first part, covering theorems 1–14, was followed by part II [19, 16–35 (1939)], containing theorems 15–30. In this paper theorems 31–50 are translated. *O. Neugebauer* (Providence, R. I.).

Saltykow, N. Correction de la solution donnée par Descartes au problème de Pappus. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. **5**, 71–74 (1939). [MF 1916]

Refers to p. 325 ff. of the 1637 edition of the "Géométrie" and corresponding theorems proved by Newton [Principia, opera ed. Horsley, vol. 2, p. 99 ff.]. At least one of the mistakes, found by the author, seems to belong to the 1927 French edition of the "Géométrie" and not to the original form. *O. Neugebauer* (Providence, R. I.).

Itard, Jean. La géométrie de Port-Royal. Enseignement Math. **38**, 27–38 (1940). [MF 2248]

***Karpinski, L. C. Bibliography of Mathematical Works Printed in America through 1850.** University of Michigan Press, Ann Arbor, Mich., 1940. xxvi+697 pp. $6.00.

Narayan, Lakshmi. Obituary notice. Ganesh Prasad. Proc. Benares Math. Soc. **1**, 107–114 (1939). [MF 1537]

Bompiani, Enrico. Gaetano Scorza. Rend. Sem. Mat. Roma **3**, 139–152 (1939). [MF 1899]

THEORY OF NUMBERS

***van Driel, M.-J. Magic Squares of $(2n+1)^2$ Cells.** Rider & Co., London, 1936. 90 pp.

***van Driel, M.-J. A Supplement to Magic Squares of $(2n+1)^2$ Cells.** Rider & Co., London, 1939. 31 pp. (Price of book and supplement: 7/6)

The author's method for constructing magic squares of order $N=2n+1$ consists essentially in finding an Eulerian square, composed of pairs of the digits $0, 1, \cdots, 2n$, and interpreting these pairs as numbers in the scale of N. [See Kraïtchik, La Mathématique des Jeux, Brussels, 1930, p. 161.] This means that every row and every column contains each of the N possible digits in both positions. If the same holds for the diagonals, the square is obviously magic; if not, it becomes magic when we suitably permute the unit-digits, or the radix-digits, or both, so as to give each diagonal the proper sum. In the case when $N=5$, the author gives a list of 18 such Eulerian squares, from which thousands more may be derived by permuting the rows or columns or both. Some of these can be constructed by the method of "uniform steps," whereby the unit-digits are placed with successive steps a to the right and b up (e.g. "Knight's moves" if $a=\pm1$ and $b=\pm2$, or vice versa), and the radix-digits with steps a' to the right and b' up. When $N=5$, the following values of (a, b, a', b') cover all possibilities: $(1, 1, -1, 1)$, the 144 squares of Bachet; $(1, 2, -1, 2)$, the 3600 pandiagonal squares; $(1, 1, -1, -2)$, the 1440 squares of De la Loubère; and $(1, 2, -1, -1)$, another set of 1440. In order to form an Eulerian square by this method, it is necessary (and sufficient) that a, b, a', b', and $ab'-a'b$ shall be prime to N. But when N is composite, other types of magic square can be made by relaxing this condition, so far as a, b, a', b' are concerned. The rows and columns will then contain repeated digits, but the proper sum may be restored by a suitable permutation of the digits. [Kraïtchik, op. cit., pp. 147–151.] The rest of the book is mainly concerned with squares having extra properties of symmetry, etc.

H. S. M. Coxeter (Toronto, Ont.).

***Candy, Albert L. Pandiagonal Magic Squares of Prime Order.** Published by the author, Lincoln, Neb., 1940. v+93 pp. $1.00.

A square matrix of N rows and columns, whose successive rows contain the numbers $1, 2, \cdots, N^2$ in natural order, is called a natural square. Any one of the $(N!)^2$ matrices derived from the natural square by permuting the rows or columns or both is called an auxiliary square. The author observes that the method of "uniform steps" [see the preceding review] is effectively equivalent to applying a linear transformation of coordinates to any auxiliary square. [See Arnoux, Les Espaces Arithmétiques Hypermagiques, Paris, 1894; and Veblen, On magic squares, Mess. of Math. **37**, 116–118 (1908).] He restricts consideration to those transformations in which the rows, columns and diagonals of the final square are all different from the rows and columns of the auxiliary square; the final square is then a "regular" pandiagonal magic square. [Rosser and Walker, Duke Math. J. **5**, 705–738 (1939); these Rev. **1**, 133.] In this manner it is easily verified that the number of such "squares of Class I" is $(N!)^2(N-3)(N-4)/8$. The new "squares of Class II" involve a further manipulation of the auxiliary square before applying the transformation. This is an ingenious interchange of pairs of numbers, which is not expressed in general terms but is worked out separately for $N=7, 11, 13, 17, 19$. The number of known pandiagonal squares is thus considerably increased; for example, when $N=7$, there are 640,120,320 of Class II, though only 38,102,400 of Class I. *H. S. M. Coxeter.*

Perez Cacho, L. On the sum of Euler functions of successive orders. Revista Mat. Hisp.-Amer. (3) **1**, 45–50 (1939). (Spanish) [MF 1674]

Let $\varphi_1(n)=\varphi(n)$ denote Euler's function, $\varphi_r(n)=\varphi(\varphi_{r-1}(n))$ its rth iteration, and m the least index for which $\varphi_m(n)=1$. The author remarks that the equality $n=\sum_{a=1}^{m}\varphi_a(n)$ holds for $n=1$; for $n=3^a$ $(a=1, 2, \cdots)$ and no other powers of a prime; for no even numbers; for $n=3p$, where p is a prime of the form $p=2^s3^e+1$, and no other numbers of the form 3^ep, where p is a prime of the form $p=2^s3^e+1$; for $n=3p$, where p is a prime of the form $p=4k+1$, if and only if it holds for k; for the products of the first d consecutive Fermat primes $(d\leqq5)$, that is, $n=3$, $3\cdot5$, $3\cdot5\cdot17$, $3\cdot5\cdot17\cdot257$, $3\cdot5\cdot17\cdot257\cdot65537$, and for no other products of different Fermat primes. *F. A. Behrend* (London).

Storchi, Edoardo. Nuova dimostrazione di un teorema sui numeri primi. Period. Mat. **19**, 274–276 (1939). [MF 904]

This paper proves the following theorem: If p is a positive odd integer and $2p+1$ is a prime, then $2^p-(-1)^{(p+1)/2}$ is divisible by $2p+1$. Euler's criterion for the factorability of the Mersenne numbers 2^p-1 is thus obtained by taking p to be an odd prime of the form $4k+3$. The desired result follows from the identity $(1\cdot3\cdot5\cdots p)2^p=2p(2p-2)(2p-4)\cdots(p+1)$. *R. D. James.*

Bang, A. S. On sums of powers of the sixth degree. Mat. Tidsskr. B. **1939**, 52–65 (1939). (Danish) [MF 714]

The author indicates three identities of type $A_1^6+\cdots+A_m^6=B_1^6+\cdots+B_n^6$, where A_i and B_i are forms in two variables. He deduces many numerical identities and, for different couples m, n, the existence of infinitely many integers representable both as sums of m and of n sixth powers. In particular, if $n\geqq 13$, there is a sixth power which is the sum of n sixth powers. *W. Feller.*

Moessner, Alfred. Zahlentheoretische Untersuchungen und Resultate. Tôhoku Math. J. **46**, 234–238 (1940). [MF 2437]

Moessner, A. Einige Diophantische Probleme und zahlentheoretische Untersuchungen. Bol. Mat. **13**, 41–47 (1940). [MF 1977]

Obláth, Richárd. Sur les nombres x^2-1. Mat. Fiz. Lapok **47**, 58–77 (1940). (Hungarian. French summary) [MF 2654]

Obláth, Richárd. Über die Zahl x^2-1. Mathematica, Zutphen. B. **8**, 161–172 (1940). [MF 1880]

The author studies the solubility of the Diophantine equation

$$x^2-1=y^n. \tag{1}$$

Using theorems of Lubelski [Prace Mat.-Fiz. **42**, 11–44 (1935)], Nagell [Norsk Vid. Selsk. Forh. **7**, 136–139 (1934)] and Vandiver [Monatsh. Math. Phys. **43**, 317–320 (1936)], the author obtains criteria for the insolubility of (1). It follows from these results that (1) has no solutions in certain cases in which the theorems of Nagell are inapplicable. In particular, the following theorem is proved. Let $p>3$ be a prime and let 2^{p-2} and 2^{p-3} be non-residues of the pth power (mod p^2). Then, if $n=p$, (1) has no solutions in integers x, y different from zero. Moreover, using a theorem of Siegel [Math. Ann. **114**, 57–68 (1937)] the author proves that (1) has at most one solution in integers different from zero. [In the proof of theorem 3 the term $+1$ is missing twice.] *A. Brauer* (Princeton, N. J.).

Johnson, L. Louise. On the Diophantine equation $x(x+1)\cdots(x+n-1)=y^k$. Amer. Math. Monthly **47**, 280–289 (1940). [MF 2241]

Let the product of n consecutive integers be

$$P_n=x(x+1)\cdots(x+n-1).$$

The author proves that $P_k=y^{2k}$ is impossible and that the equation $P_n=y^k$, for $n=4$, 5, 6, 7 and $k>7$ prime, has at most one solution. The proof of the first result is elementary and uses a result of Sylvester and Schur. The proof of the second result is based on a deep theorem of Siegel. *P. Erdös* (Princeton, N. J.).

Pillai, S. S. On m consecutive integers. II. Proc. Indian Acad. Sci., Sect. A. **11**, 73–80 (1940). [MF 1858]

The greater part of this paper is devoted to the proof of the fact that the equation

$$n(n+1)\cdots(n+m-1)=y^r \tag{1}$$

has no solutions n and y for the case $2\leqq m\leqq 16$, $r\geqq(m+3)/2$. The proof depends on an interesting theorem proved in the first communication [in the same vol., pp. 6–12; these Rev. **1**, 199]. This states that for $m\leqq 16$, in every set of m consecutive integers, there is at least one which is prime to every other integer in the set. Two further theorems are given with outlines of their proofs. The first states that, for $r\geqq 3$, the equation (1) has at most a finite number of solutions. The second states that there is a number $c(m)$ such that (1) has no solution for $r>c(m)$. Here the author clearly intends that the condition $m\geqq 2$ shall hold. The proofs are accomplished by making use of theorems of Thue and Siegel. *H. S. Zuckerman* (Seattle, Wash.).

Bell, E. T. Compound multiplicative Diophantine systems. Proc. Nat. Acad. Sci. U. S. A. **26**, 462–466 (1940). [MF 2475]

The author considers the system of simultaneous Diophantine equations $\sum_{p=1}^{i}a_p(x,a,n)_p=0$, $\sum_{q=1}^{j}b_q(y,b,m)_q=0,\cdots$, $\sum_{r=1}^{k}c_r(z,c,l)_r=0$, where (x,a,n) denotes the product $x_{i_1}^{a_{i_1}}\cdots x_{i_{n_i}}^{a_{i_{n_i}}}$, $a, b, \cdots, c$ are constant integers and the $x, y, \cdots, z$ are the unknowns. The definition of this compound multiplicative system is given in terms of the simple multiplicative system already studied by the author [Amer. J. Math. **55**, 50–66 (1933)]. It is proved that the problem of finding the complete solution of the above compound system is reduced to that of a simple system and that the solution of the compound system may be written as polynomials with rational integral coefficients in a certain minimum number of parameters. *I. A. Barnett.*

Fogels, E. Über die Möglichkeit diophantischer Gleichungen in relativ quadratischen Zahlenkörpern. Acta Univ. Latviensis **3**, 273–284 (1940) = Publ. Sem. Math. Univ. Lettonie, no. 15 (1940). (German. Latvian summary) [MF 2149]

Let $f(x, y, z, \cdots)=0$ be a Diophantine equation of degree three with coefficients in the field K. The author proves some theorems on the solubility of $f=0$ in quadratic fields over K. For instance, he gives a new proof for the following theorem of Nagell [Ark. Mat. Astr. Fys. **25**, no. 5 (1935)]. If the equation $x^3+y^3+a=0$ with $a\neq 0$ has a solution in $K(\sqrt{-3})$, then there exists a solution in K too. Theorems 7 and 8 are not quite correct; this is shown by the following example: $f=x^3+2y^3-ax=0$, $x=y=X-Y=g(X, Y)$, $f[g(X, Y)]=0$ for $X=Y$. *A. Brauer* (Princeton, N. J.).

Krasner, Marc. Sur le théorème de Fermat. C. R. Acad. Sci. Paris **210**, 92–94 (1940). [MF 1240]

In a paper [C. R. Acad. Sci. Paris **208**, 1468–1471 (1939)] the author has proved that the equation $x^p+y^p+z^p=0$, the class number of $K(\rho)$, $\rho=2^{1/p}$, being odd, has a rational solution, $xyz\neq 0$, z even, $x\not\equiv y \pmod p$, if and only if there exist an even number z and an odd number M so that $\alpha=z^2-\rho^2M$ is the square of a number of $K(\rho)$. The suppositions z even and $x\not\equiv y \pmod p$ were only used in that part of the demonstration in which the residue symbol of Hilbert $(\alpha, 1-\rho\,|\,(\rho))_2$ was proved to have the value -1. The author now shows that in the case z odd, $x\not\equiv y \pmod p$, from the hypothesis $M\not\equiv 0 \pmod 4$ also follows the value -1 for this symbol. *N. G. W. H. Beeger* (Amsterdam).

Krasner, M. À propos du critère de Sophie Germain-Furtwängler pour le premier cas du théorème de Fermat. Mathematica, Cluj **16**, 109–114 (1940). [MF 2492]

The author makes use of an idea due to Sophie Germain to prove that the Fermat equation $x^p+y^p+z^p=0$ cannot be satisfied by integers x, y, z which are all non-divisible by p, if there exists a prime q of the form $q=up+1$, where u is not a multiple of 3 and $3^{u/4}<q$, $2^u\not\equiv 1 \pmod q$. The proof uses a theorem of Furtwängler but is otherwise "elementary." He concludes that we may expect to be able to prove the first

case of Fermat's theorem when the theory of primes has been sufficiently developed. *H. W. Brinkmann.*

Rosser, Barkley. A new lower bound for the exponent in the first case of Fermat's last theorem. Bull. Amer. Math. Soc. **46**, 299–304 (1940). [MF 1837]

In the author's notation an odd prime p is called improper if there are integers α, β and γ in the field of the pth roots of unity such that $\alpha\beta\gamma$ is prime to p and $\alpha^p+\beta^p+\gamma^p=0$. It was proved in a previous paper [Bull. Amer. Math. Soc. **45**, 636–640 (1939); cf. these Rev. **1**, 5] that there are no improper primes less than 8,332,403. The proof was based on the following theorem of Morishima: If p is an improper prime then, for each prime $m\leqq 31$, we have $m^{p-1}\equiv 1 \pmod{p^2}$. In the present paper the fact that p must exceed 8,000,000 is used to show that the theorem of Morishima holds also when $m=37$ or 41. From this it follows that there are no improper primes less than 41,000,000. The process could be repeated with the possibility of showing that the theorem of Morishima holds for all primes $m\leqq N_1$, where $N_1\geqq 43$, thus obtaining a still higher bound. However, an argument is advanced which appears to rule out the possibility of obtaining an indefinitely large bound by this method. *R. D. James* (Saskatoon, Sask.).

Bauer, Michael. Über zusammengesetzte relativ Galoissche Zahlkörper. Acta Litt. Sci. Szeged **9**, 206–211 (1940). [MF 1223]

Bauer, Michael. Über die Zusammensetzung algebraischer Zahlkörper. Acta Litt. Sci. Szeged **9**, 212–217 (1940). [MF 1224]

The theorems proved in these two papers are reported in the reviews of the Hungarian versions [these Rev. **1**, 135]. They deal with degree and ramification order of prime ideals in composite fields. The proofs are strictly group-theoretic in the following sense. According to the Dedekind-Hilbert theory, relations between the ideals in question correspond to relations between certain subgroups of the Galois group (groups of decomposition, inertia, and ramification). Making use of standard properties of these groups the author obtains his results by studying orders and indices of subgroups.

M. A. Zorn (Los Angeles, Calif.).

Gelfond, A. Sur la divisibilité de la différence des puissances de deux nombres entiers par une puissance d'un idéal premier. Rec. Math. [Mat. Sbornik] N.S. **7** (**49**), 7–25 (1940). (French. Russian summary) [MF 2277]

The author makes use of $\mathfrak{p}$-adic analysis to study the expression $\alpha^n\pm\beta^m$, where α, β are numbers in an algebraic field R such that $\alpha^n\neq\beta^m$ for any rational integers m, n with $mn\neq 0$. Thus he proves that, if $\mathfrak{p}$ is a prime ideal of R not contained in the numerator and denominator of α and β, then for any $\epsilon>0$ there exists a $n_0=n_0(\alpha,\beta,\mathfrak{p},\epsilon)$ such that $\alpha^n\pm\beta^m$ with $n>n_0$, $n\geqq m>0$ is not divisible by $\mathfrak{p}^\gamma$ if $\gamma>\log^{2+\epsilon} n$. Using this result and a previous one [cf. these Rev. **1**, 295] the author proves that the equation $\alpha^x+\beta^y=\gamma^z$ (α, β, γ are in R, none equals 0, ± 1 and at least one of them is not a unit) has at most a finite number of integral solutions x, y, z except when $\alpha=\pm 2^{n_1}$, $\beta=\pm 2^{n_2}$, $\gamma=\pm 2^{n_3}$ (n_1, n_2, n_3 rational). *H. W. Brinkmann.*

Pall, Gordon. An almost universal form. Bull. Amer. Math. Soc. **46**, 291 (1940). [MF 1835]

The author studies the form $h=(1, 2, 7, 13)$ and shows that it represents all positive integers with the exception of 5. This completes the count (88 in all) of all positive quaternary forms without cross products which represent all integers with but one exception, carried out by P. R. Halmos [Bull. Amer. Math. Soc. **44**, 141–144 (1938)].

A. E. Ross (St. Louis, Mo.).

***Schnirelman, L. G. Prime Numbers.** Moscow-Leningrad, 1940. 60 pp. (Russian)

Elementary exposition of foundations of the theory of prime numbers.

Broderick, T. S. On proving certain properties of the primes by means of the methods of pure number theory. Proc. Roy. Irish Acad. Sect. A. **46**, 17–24 (1940). [MF 1800]

This is a continuation of the author's attempts to state and prove elementary theorems on the distribution of primes in the language of "pure number theory," using only elementary properties of rational numbers, that is, essentially of integers. [Cf. J. London Math. Soc. **14**, 303–310 (1939); these Rev. **1**, 41.] The main results are

$$\sum_{p\leqq n}\frac{\lambda(p)}{p}=\lambda(n)+O(\lambda_2(n)),\quad \sum_{p\leqq n}\frac{1}{p}=\sum_{r\leqq n}\frac{1}{r\lambda(r)}+O(1),$$

where, for rational $x\geqq 1$,

$$\lambda(x)=\sum_{r\leqq x} r^{-1},\quad \lambda_2(x)=\lambda(\lambda(x)).$$

By the methods of classical mathematics (involving the function "log") the right hand side of the second result can be replaced by $\lambda_2(n)+O(1)$, but the author records that by the methods of "pure number theory" he is able to obtain only the weaker result

$$\sum_{p\leqq n}\frac{1}{p}\sim\lambda_2(n).$$

A. E. Ingham (Berkeley, Calif.).

Erdös, P. The difference of consecutive primes. Duke Math. J. **6**, 438–441 (1940). [MF 2328]

Let p_n be the nth prime. Using the Riemann hypothesis, Hardy and Littlewood proved some years ago that

$$A=\liminf_{n\to\infty}\frac{p_{n+1}-p_n}{\log p_n}\leqq\frac{2}{3},$$

and Rankin recently proved, again by using the Riemann hypothesis, that $A\leqq 3/5$. Their proofs have not been published. The author proves, without the Riemann hypothesis, that $A<1-c$ for a certain $c>0$. Moreover, he conjectures the following theorem: Let n be any integer and let $0<a_1<a_2<\cdots<a_{\varphi(n)}<n$ be the $\varphi(n)$ integers relatively prime to n, then

$$\sum_{r=1}^{\varphi(n)}(a_{r+1}-a_r)^2<k\frac{n^2}{\varphi(n)},$$

where k is independent of n. *A. Brauer.*

Rankin, R. A. The difference between consecutive prime numbers. II. Proc. Cambridge Philos. Soc. **36**, 255–266 (1940). [MF 2527]

In this paper the author proves the theorem: Let θ be the lower bound of all positive numbers σ_0 such that no Dirichlet L-function $L(s,\chi)$ has a zero at $s=\sigma+it$, $\sigma>\sigma_0$. Then for every $\epsilon>0$ there exists a positive number $N_0=N_0(\epsilon)$ such that for all $N\geqq N_0$ there is always a pair of primes p, p' satis-

fying the relations

$$N<p'<p\leqq 2N, \quad p-p'<\left(\frac{1+4\theta}{5}+\epsilon\right)\log p.$$

The proof proceeds along the following lines. The integral

$$I(N,k)=\int_0^1 |S(\alpha)\Psi(\alpha)|^2 d\alpha,$$

where

$$S(\alpha)=\sum_{N<p\leqq 2N} e^{2\pi i\alpha p}\log p, \quad \Psi(\alpha)=\sum_{\nu=1}^{k} e^{2\pi i\nu\alpha},$$

involves the sum

$$\sum_{\nu=1}^{k-1}(k-\nu)w(\nu),$$

where $w(\nu)$ is the number of prime pairs p, p' satisfying $N<p'<p\leqq 2N$ and $p-p'=\nu$. Since the integrand is positive, $I(N,k)$ obviously exceeds the integral taken over certain parts of the interval (0, 1) called major arcs. The assumption that $w(\nu)=0$ for $1\leqq\nu\leqq k-1$ then leads to a contradiction when

$$k=\left[\left(\frac{1+4\theta}{5}+\epsilon\right)\log N\right]+1.$$

R. D. James (Saskatoon, Sask.).

Pillai, S. S. Generalisation of a theorem of Mangoldt. Proc. Indian Acad. Sci., Sect. A. **11**, 13–20 (1940). [MF 1499]

Let $N_{r,t}(x)$ denote the number of numbers not exceeding x, whose number of prime factors (each counted according to its multiplicity) belongs to the residue class t mod r, and $N'_{r,t}(x)$ the number of such squarefree numbers. Then

$$(1) \qquad N_{r,t}(x)\sim\frac{x}{r},$$

$$(2) \qquad N'_{r,t}(x)\sim\frac{6}{\pi^2}\frac{x}{r}.$$

The author's proof holds for primes r only; but in an appendix the result is proved for all cases by an argument due to Chowla. (1) is a generalization of von Mangoldt's theorem

$$L(x)=\sum_{n\leqq x}(-1)^{\rho(n)}=o(x)$$

($\rho(n)$: the number of prime factors of n), which is equivalent to $N_{2,0}(x)\sim N_{2,1}(x)\sim x/2$; von Mangoldt also proved that $N'_{2,0}(x)\sim N'_{2,1}(x)$. A different proof of (2) is due to S. Selberg [Math. Z. **44**, 306–316 (1938)]. *F. A. Behrend.*

Vinogradow, I. Distribution of primes of an arithmetical progression to a given modulus. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 27–36 (1940). (Russian. English summary) [MF 1980]

It is proved that, if $N>2$, $0<q\leqq N$, $0\leqq L<Q$, $(Q,L)=1$, $(Q,q)=1$, then

$$(1) \qquad \sum_{\substack{N-A<p\leqq N\\ p\equiv L \pmod Q}} e^{2\pi i\alpha p}=O(R), \quad \alpha=\frac{a}{q};\ (a,q)=1;\ 1\leqq A\leqq N,$$

$$(2) \qquad T_1-\sigma T=O(R), \qquad 0<\sigma\leqq 1;\ Q\leqq A\leqq N,$$

where T is the number of primes p in the range of summation in (1), T_1 is the number of these p for which $0\leqq p<\sigma q$ (mod q),

$$R=\frac{A}{Q}N^{\epsilon}\left(\frac{1}{q}+\frac{N^{\frac{1}{2}}Q}{A}+\frac{NQ^2q}{A^2}\right)^{\frac{1}{2}},$$

and the constants of the O's depend only on $\epsilon(>0)$. The proof of (1), from which (2) is deduced, is by the methods used in the author's previous estimations of trigonometrical sums involving primes. *A. E. Ingham.*

Vinogradow, I. Simplest trigonometrical sums with primes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **23**, 615–617 (1939). [MF 1952]

For the sum

$$S=\sum_{N-A<p\leqq N} e^{2\pi i(a/q)p^n}, \quad q>0;\ (a,q)=1;\ n \text{ a positive integer},$$

the author states the following estimates ("near to the best possible ones"), in which $N>2$, $r=\log N$, and $c, \sigma, \alpha, \epsilon, \eta, h$ are positive constants, of which ϵ, η, h are arbitrarily small:

$S=O(Ar^{2c-1}q^{\tau-\frac{1}{2}})$ if $cNe^{-r^{1-2\epsilon}}\leqq A\leqq N$, $0<q\leqq e^{r^{\epsilon}}$,

$S=O(Aq^{\tau-\frac{1}{2}})$ if $N^{1-\alpha}\leqq A\leqq N$, $e^{r^{\epsilon}}<q\leqq N^{\sigma}$, $4\alpha+6\sigma=\dfrac{1}{1+h}$,

$S=O(Aq^{\tau-\frac{1}{2}})$ if $N^{1-\alpha}\leqq A\leqq N$, $N^{\sigma}<q\leqq N^{\sigma}$, $3\alpha+5\sigma=\dfrac{1}{1+h}$.

It is asserted that, when $n=1$, the second and third results hold under wider conditions. The author also announces improved forms of his previous estimates of

$$\sum_{x=Q+1}^{Q+P} e^{2\pi i mf(x)} \quad \text{and} \quad \sum_{x=Q+1}^{Q+P} e^{2\pi iF(x)},$$

where m is a positive integer, $f(x)$ a real polynomial and $F(x)$ a real function whose nth and $(n+1)$th derivatives satisfy certain inequalities. *A. E. Ingham.*

Fogels, E. On average values of arithmetical functions. Acta Univ. Latviensis **3**, 285–313 (1940)=Publ. Sem. Math. Univ. Lettonie, no. 16 (1940). (English. Latvian summary) [MF 2148]

For a given arithmetic function $a(n)$ the writer considers the problem of determining the least $h=h(x)$ such that $\sum_{n\leqq x}a(n)\sim\psi(x)$ will imply

$$\sum_{x<n\leqq x+h} a(n)\sim\psi(x+h)-\psi(x).$$

Using methods of Heilbronn and Ingham he proves that

$$\sum_{x<n\leqq x+h}\Lambda(n)\sim h,$$

where $h=x^{\theta}$, $\Theta=(1+4c/2+4c)+\epsilon$, and c is such that $\zeta(\frac{1}{2}+it)=O(t^c)$; the best known value of $c=19/116$ gives $\Theta=(48/77)+\epsilon$. From this result it follows that there is at least one prime in the interval y^{ϑ}, $(y+1)^{\vartheta}$ $(y>y_0)$, where $\vartheta=1/(1-\Theta)$ (we may take $\vartheta=(77/29)+\epsilon$). Similar results are derived for various other arithmetic functions.

L. Carlitz (Durham, N. C.).

Brun, Viggo. Deux transformations élémentaires de la fonction zeta de Riemann. Revista Ci., Lima **41**, 517–525 (1939). [MF 1644]

The author establishes the formula

$$\zeta(s)=1+\frac{1}{s-1}-\beta(s), \qquad s \text{ real}, \ s>1,$$

where $\zeta(s)$ is the Riemann zeta-function and

$$\beta(s)=\sum_{m=1}^{\infty}\frac{1}{2^m}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2^m}{2^m+n}\right)^s.$$

The function $\beta(s)$ may be written in other forms, one of these being a double series which converges also for $s=-1, -2, \cdots$. The proof uses identities of which the following is an example:

$$\sum_{n=1}^{2^r}(-1)^{n-1}n^{-s}=\sum_{n=1}^{2^r}n^{-s}-2^{1-s}\sum_{n=1}^{2^{r-1}}n^{-s}.$$

R. D. James (Saskatoon, Sask.).

Wintner, Aurel. Riemann's hypothesis and almost periodic behavior. Revista Ci., Lima **41**, 575–585 (1939). [MF 1648]

The author gives the following necessary and sufficient conditions that the Riemannian hypothesis be true; namely, that the reciprocal of the zeta function be a Besicovitch almost periodic function of infinite order (B^∞) on every vertical line in the right hand half of the critical strip (when $\sigma>\frac{1}{2}$). An equivalent condition is that every negative integral power of the zeta function be of class B^2 when $\sigma>\frac{1}{2}$. He also shows that the corresponding condition for positive integral powers of the zeta function is equivalent to the Lindelöf hypothesis. *R. H. Cameron.*

Crum, M. M. On some Dirichlet series. J. London Math. Soc. **15**, 10–15 (1940). [MF 2578]

The author discusses formal properties of some Dirichlet series $\sum f(n)n^{-s}$ for which he either proves or assumes that they are expressible as products and quotients of the zeta-function with arguments $\alpha s+\beta$. In several cases $f(n)$ is constructed with the function $\sigma_\lambda^{(\mu)}(n)$ which is the sum of the λth powers of those divisors of n which are μth powers of integers. *S. Bochner* (Princeton, N. J.).

Petersson, Hans. Konstruktion der sämtlichen Lösungen einer Riemannschen Funktionalgleichung durch Dirichlet-Reihen mit Eulerscher Produktentwicklung. II. Math. Ann. **117**, 39–64 (1939). [MF 1382]

This paper and the previous Part I [Math. Ann. **116**, 401–412 (1939)] extend the work of Hecke [Math. Ann. **114**, 1–28, 316–351 (1937)] on Dirichlet series $D(s)=\sum_1^\infty a(n)n^{-s}$ associated with integral modular forms $f(\tau)=\sum_0^\infty a(n)e^{2\pi i n\tau/Q}$ of "type $(-k, Q)$" (that is, dimension $-k$ and Stufe Q; k and Q positive integers). The significance of the title is that the transformation formulae for the $f(\tau)$ correspond to functional equations, or systems of functional equations, for the analytic functions defined by the $D(s)$, and that attention is focused on those $D(s)$ which have Euler products. In Part I it was proved that, when $Q=1$, the linear set of all $D(s)$ has a (linearly independent) basis $D_1(s), \cdots, D_s(s)$, and only one (except for order), in which each $D_i(s)$ has an Euler product; or, what is the same thing after Hecke, that the linear set of all $f(\tau)$ has a basis $f_1(\tau), \cdots, f_s(\tau)$, and only one (apart from order and constant factors), made up of eigenfunctions of a certain commutative ring of linear operators introduced by Hecke. Here, in Part II, the more complicated case $Q>1$ is discussed. The author considers the linear set $S(t, \epsilon, Q)$ of cusp-forms $f(\tau)$ associated in the manner defined by Hecke with a given divisor t of Q and a given character $\epsilon(n)$ mod Q (the "complementary" set $E(t, \epsilon, Q)$ of Eisenstein series having already been analyzed by Hecke), and shows that results analogous to the above hold in $S(t, \epsilon, Q)$ and in the associated set of $D(s)$ if $t=1$ or if every prime factor of t divides Q/t. The proof consists in finding a normal orthogonal basis of $S(t, \epsilon, Q)$ in the metric introduced by the author in Part I. If t contains a prime factor which does not divide Q/t, a complete reduction in the sense of the above theorems is not always possible, and the failure is analyzed in detail in the case $Q=q$, an odd prime. On the basis of this discussion a generalization of Ramanujan's conjecture $|\tau(n)|\leqq n^{11/2}d(n)$ for the coefficients in $x\{(1-x)(1-x^2)\cdots\}^{24}=\sum_1^\infty\tau(n)x^n$ is proposed, and an analogy with the Riemann hypothesis for congruence zeta-functions is indicated. *A. E. Ingham.*

Zuckerman, Herbert S. The computation of the smaller coefficients of $J(\tau)$. Bull. Amer. Math. Soc. **45**, 917–919 (1939). [MF 782]

Ramanujan's theorem about the divisibility of $p(25n+24)$ is connected with the multiplicator equation of $J(\tau)$ of the 5th order. The author makes use of this fact and finds $J(\tau)$ expressed by

$$\sum_{n=0}^{\infty}p(25n+24)e^{2\pi i n\tau}$$

and some other modular functions with arithmetically simple coefficients. This enables him to utilize Gupta's table of $p(n)$ for n up to 600 [Proc. London Math. Soc. **39**, 142–149 (1935); **42**, 546–549 (1937)] to compute the first 24 coefficients of $J(\tau)$, the first seven of which agree with values given by Berwick [Quart. J. Math. **47**, 94–103 (1916)]. *H. Rademacher* (Swarthmore, Pa.).

Jarník, Vojtěch. Eine Bemerkung zur Gitterpunktlehre. Časopis Pěst. Mat. Fys. **69**, 57–60 (1940). (German. Czech summary) [MF 1940]

Let S_r be the set of points $a=(a^{(1)}, \cdots, a^{(r)})$ $(a^{(j)}>0)$, $A_a(x)$ the number of lattice points in the ellipsoid $a^{(1)}u_1^2+\cdots+a^{(r)}u_r^2\leqq x$, $V_a(x)$ the volume of this ellipsoid, and

$$P_a(x)=A_a(x)-V_a(x), \quad M_a(x)=\int_0^x P_a(y)dy.$$

It is known that, when $x\to\infty$,

$$\overline{\lim}\frac{P_a(x)}{x^{\frac{1}{2}r-1}}<+\infty, \quad \underline{\lim}\frac{P_a(x)}{x^{\frac{1}{2}r-1}}>-\infty, \qquad r>4,$$

$$\overline{\lim}\frac{M_a(x)}{x^{r-1}}<+\infty, \quad \underline{\lim}\frac{M_a(x)}{x^{\frac{1}{2}r+1}}>0, \text{ and}$$

$$\overline{\lim}\frac{M_a(x)}{x^{\frac{1}{2}r+1}(\log x)^{2r+2}}<+\infty, \qquad r\geqq 4,$$

the last for almost all a, the others for all a, in S_r. It is shown here, with the help of continuity properties of $P_a(x)$ and $M_a(x)$ in conjunction with known results for rational points a, that, given any $f(x)\to\infty$, we have

$$\overline{\lim}\frac{P_a(x)}{x^{\frac{1}{2}r-1}}f(x)=+\infty, \quad \underline{\lim}\frac{P_a(x)}{x^{\frac{1}{2}r-1}}f(x)=-\infty, \qquad r>4,$$

$$\overline{\lim}\frac{M_a(x)}{x^{r-1}}f(x)=+\infty, \quad \underline{\lim}\frac{M_a(x)}{x^{\frac{1}{2}r+1}(\log x)^{2r+2}f(x)}=0, \qquad r\geqq 4,$$

except in a set of the first category in S_r.

A. E. Ingham (Berkeley, Calif.).

Davenport, H. Note on linear fractional substitutions with large determinant. Ann. of Math. (2) **41**, 59–62 (1940). [MF 1005]

If the integer N is sufficiently large and if P is prime to N, then for any two intervals I, I' of length $N^{(5/6)+\epsilon}$ there exist integers α and β', respectively contained in I and I', such that $\alpha\beta' \equiv P \pmod N$. The proof of this lemma is "an exercise in the use of exponential sums." The number of solutions is first represented in the form $1/N^2$ times a sum of roots of unity. This expression is then decomposed into four parts, the first of which is asymptotically $\varphi(N)N^{-(1/2)+\epsilon}$ (trivial). The lemma will certainly be true if the three remaining parts are of lower order; this is proved on the basis of the following O-relations (letting $e(t)=\exp(2\pi it/N)$): (a) the Kloosterman sums $\sum e(au+bu^{-1})$ are $O(N^{(2/3)+\epsilon}(a, N))$, where $u=1, \cdots, N-1$; (b) for $x=1, \cdots, N-1$, $x'=\min(x, N-x)$, the sums (y in any interval) $\sum e(xy)$ are $O(N/x')$.

If $z_0(\Re z_0=r, |z_0|^2=m)$ is a complex number not equal to zero, then it may be approximated by fixed points $f(\Re f=(\alpha-\beta')/2\beta, |f|^2=-\alpha'/\beta)$ of linear substitutions $Z=(\alpha z+\alpha')/(\beta z+\beta')$, with integral coefficients of determinant P, such that (a) both $\Re f-\Re z_0$ and $|z_0|^2-|f|^2$ are $O(P^{-(1/12)+\epsilon})$, (b) the signs of these two differences may be preassigned arbitrarily.

The construction is such that β is chosen first, in dependence on P; afterwards the existence of the numbers α and β' is shown. With $k>m$ fixed, β is taken such that $(\beta, P)=1$, $P=k\beta^2+O(\beta)$; the conditions to be fulfilled by α and β' are: (a) $\alpha-\beta'=2r\beta+O(\beta^{(5/6)+\epsilon})$, (b) $\alpha\beta'=(k-m)\beta^2+O(\beta^{(11/6)+\epsilon})$, (c) $\alpha\beta' \equiv P \pmod\beta$. The author provides explicitly two intervals I, I' of length $\beta^{(5/6)+\epsilon}$ such that the first two conditions are satisfied whenever α, β' are taken from I, I'; the lemma guarantees that one of these pairs satisfies also the last condition.

M. A. Zorn (Los Angeles, Calif.).

Mahler, Kurt. On a geometrical representation of p-adic numbers. Ann. of Math. (2) **41**, 8–56 (1940). [MF 1003]

A geometric representation of a p-adic integer by means of a sequence $z_n=x_n+iy_n$ is obtained in the following manner. For any r(esidue) c(lass) $\zeta \bmod P^n$ there exist integer matrices $T_n=(p_n, p_n'; q_n, q_n')$ such that (a) $(1, \zeta)T_n \equiv (0, 0) \bmod P^n$; (b) the determinant of T_n is P^n; (c) the elements q_n, q_n' of the second row are relatively prime. The last condition ensures that the r.c. mod P^n represented by T_n is unique; also, if the r.c. of T_{n+1} is contained in the r.c. of T_n, we have $T_{n+1}=T_n\Omega$, where Ω is an integer matrix. In particular, if two matrices represent the same r.c., the factor Ω will belong to the modular group Γ; and if T_n is a sequence representing the r.c. of a fixed number ζ, then the quotients $T_n^{-1}T_{n+1}$ will have determinant P. Choosing a number λ in the classical fundamental domain F of Γ, we normalize the matrices of the sequence by asking that $T_n(z_n)=\lambda$ define numbers z_n in the fundamental domain F; and this sequence of complex numbers is the geometric representation of ζ. Now let $\Phi(X, Y)=2(X-\lambda Y)(X-\bar\lambda Y)/|\lambda-\bar\lambda|$ and consider the transforms $\Phi((X, Y)T_n)$; computing the coefficient of X^2 in two ways we obtain the equation $\Phi(p_n, q_n)=P^n/y_n$. The numbers y_n (and z_n) therefore provide information about the "size" of the pairs p_n, q_n which according to the above condition (a) solve the congruence $u+v\zeta \equiv 0 \bmod P^n$. For instance, since z_n is in F, $y_n \geqq 3^{1/2}/2$, we get: For all n this congruence has solutions p_n, q_n with $0<\Phi(p_n, q_n) \leqq P^n 2\cdot 3^{-1/2}$ (shorter: $u+v\zeta$ satisfies $0<\Phi \leqq P^n 2\cdot 3^{-1/2}$). The matrices T_n furnish the best solutions, that is, $\Phi(p, q)<\Phi(p_n, q_n)$ is impossible, equality holds only in specified cases (a "Lagrange" type theorem). It is shown that y_n tends towards infinity exactly for rational ζ; if ζ is irrational, $y_n \leqq P^{1/2}$ infinitely often, and it may almost always be greater than $P^{1/2}-\epsilon$. Consequence: if ζ is irrational, then for an infinity of indices $u+v\zeta$ does not satisfy $0<\Phi<P^nP^{-1/2}$; but for every $\epsilon>0$ there exists a ζ such that for all large indices $0<\phi \leqq (P^{-1/2}+\epsilon)P^n$ is satisfied ("Khintchine"). On the same basis: If ζ is an irrational and θ an arbitrary p-adic integer, then $u+v\zeta+\theta$ satisfies infinitely often $\phi<P^n(P+1)/4P^{1/2}$ ("Tchebycheff").

A group of results is obtained (by force) for individual primes P. For $P=2, 3$ one out of every three, for $P=5$ one of every two subsequent $y_n \geqq 1/c_P$ (this number is $7^{1/2}/2$, $2^{1/2}$, 1, respectively). This implies: For every ζ, and at least one of three subsequent indices, $u+v\zeta$ satisfies $0<\phi \leqq c_P P^n$; and for $\epsilon>0$ there exists a ζ, such that for large indices $0<\phi \leqq (c_P-\epsilon)P^n$ is impossible ("Hurwitz-Borel"). The same theorem holds, with $c_P=2/3^{1/2}$, for all primes $P \equiv 1 \bmod 6$.

M. A. Zorn (Los Angeles, Calif.).

Gelfond, A. Sur l'approximation du rapport des logarithmes de deux nombres algébriques au moyen de nombres algébriques. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1939**, 509–518 (1939). (Russian. French summary) [MF 2093]

It is proved that, if α, β, θ are algebraic numbers with $\alpha \neq 0, 1$, $\beta \neq 0, 1$, $\log\alpha/\log\beta$ irrational, and $H_0\theta^n+H_1\theta^{n-1}+\cdots+H_n=0$, H_i rational integers, $\max|H_i| \leqq H$, then

$$\left|\frac{\log\alpha}{\log\beta}-\theta\right| > e^{-(\log H)^{2+\epsilon}}$$

for any $\epsilon>0$ and all sufficiently large H. This is a sharpened form of the theorem that β^θ cannot be algebraic if β, θ are algebraic with $\beta \neq 0, 1$, and θ irrational [A. Gelfond, Bull. Acad. Sci. URSS. Sér. Math. **1934**, 623–630 (1934); T. Schneider, J. Reine Angew. Math. **172**, 65–69 (1934)], and the proof is a development of the author's proof of the original theorem. It is deduced that (apart from trivial cases of exception not specified by the author) an equation of the form

$$\alpha^x+\sum_{k=1}^{m_1} P_k(x, y)\alpha_k^y=\beta^y+\sum_{l=1}^{m_2} Q_l(x, y)\beta_l^y,$$

where $\alpha, \alpha_k, \beta, \beta_l$ are algebraic, $|\alpha|>|\alpha_k|$, $|\beta|>|\beta_l|$, and P_k, Q_l are polynomials with integral algebraic coefficients, can have at most a finite number of solutions in rational integers x, y.

A. E. Ingham (Berkeley, Calif.).

Segal, B. Approximation of complex numbers by a sum of powers of integers with a given complex exponent. Rec. Math. [Mat. Sbornik] N.S. **5** (**47**), 147–184 (1939). (Russian. English summary) [MF 1427]

". . . The main result of the present paper may be stated in the form of the following theorem: Let $a+bi$ be a given complex number, where $a>1$ and $b \neq 0$, $n=[a]+2$, $a'=n-a-1$, $r \geqq r_0$ an integer, where

$$r_0=[2n/\rho_n]+1=(2n/\rho_n)+\epsilon', \qquad 0<\epsilon' \leqq 1;$$

ρ_n depends only on n, and $0<\rho_n \leqq 1/n$. Further, let $N=N_1+N_2i$ be any complex number, $X=|N|^{1/a}$, $\Delta_1=X^{-a}$, $\alpha=a'\epsilon'/r_0$, I_N the number of representations of N in the

form

$$N_1+N_2i=h_1+h_2i+\sum_{k=1}^{r}x_k^{\alpha+bi},$$

where $x_1, \cdots, x_r$ are integers satisfying the condition $0<x_k<X$, and h_1, h_2 satisfy the conditions $-\Delta_1 \leqq h_1, h_2 \leqq \Delta_1$. Then we have the asymptotic formula

$$I_N=LX^{r-2a-2\alpha}(1+O(X^{-\alpha})),$$

where L exceeds a positive number depending only on a, b and r." The value of ρ_n is connected with the estimation of trigonometrical sums of the form

$$\sum_{A\leq X\leq B}\exp(2\pi i f(x)).$$

We can put

$$r_0=\begin{cases}n\cdot 2^{n+1}+1 & \text{if } 1<a<14,\\ [8n^4\log n]+1 & \text{if } a\geqq 14.\end{cases}$$

A less exact result was published by the author in a note in C. R. (Doklady) Acad. Sci. URSS (N.S.) **19**, 667–670 (1938). *K. Mahler* (Manchester).

Raikov, D. A. A proof of a theorem of L. G. Schnirelmann concerning the density of the arithmetic sum of sets. Uspekhi Matem. Nauk **7**, 97–101 (1940). (Russian) [MF 2171]

Let A and B be two sets of non-negative real numbers. By the arithmetic sum $A+B$ one understands the set of all numbers which can be written in the form $a+b$, where $a\in A$ and $b\in B$. The density of a set E on the interval $(0, x)$ is defined as

$$\pi_E(x)=\text{gr.l.b.}_{0<\xi<x}\frac{m_E(0,\xi)}{\xi},$$

where $m_E(0, \xi)$ denotes the inner measure of the common part of E and the interval $(0, \xi)$. Generalizing a result of Schnirelman [Rec. Math. [Mat. Sbornik] N.S. **5** (**47**), 211–215 (1939); these Rev. **1**, 202, where the theorem was stated without proof], the author shows that for every two sets A and B of non-negative numbers containing 0 as their common point and for every $x>0$

$$\pi_{A+B}(x)\geqq\min\{\pi_A(x)+\pi_B(x), 1\}.$$

The proof proceeds by considering first the case when A and B are sums (in the point-set theoretic sense) of a finite number of intervals. *M. Kac* (Ithaca, N. Y.).

Raikov, D. On the addition of point-sets in the sense of Schnirelmann. Rec. Math. [Mat. Sbornik] N.S. **5** (**47**), 425–440 (1939). (Russian. English summary) [MF 2310]

[For the first part of this paper, cf. the preceding review.] The author calls a set B "a weak base of order k for the interval $(0, y)$" if kB $(=B+\cdots+B$ (k times), the sum is meant in the arithmetic sense) is everywhere dense in $(0, y)$ and k is the smallest integer with the above property. The following point-set theoretic analogue of a well-known theorem of Erdös is proved. If B is a weak base of order l for $(0, y)$ and A an arbitrary set in $(0, y)$ whose density is not less than α, then the density of the arithmetic sum $A+B$ is not less than $\alpha+(\alpha(1-\alpha)/2l)$. Theorems on arithmetic sums mod 1 are also given. *M. Kac* (Ithaca, N. Y.).

ANALYSIS

Scott, W. T. and Wall, H. S. Continued fraction expansions for arbitrary power series. Ann. of Math. (2) **41**, 328–349 (1940). [MF 1816]

Corresponding to every power series (1) $1+\sum c_n x^n$ there exists a unique continued fraction (2) $1+K[a_n x^{\alpha_n}/1]$, where the α_n are positive integers, and conversely [Leighton and Scott, Bull. Amer. Math. Soc. **45**, 596–605 (1939); these Rev. **1**, 7]. The authors find a new and useful method for determining the complex constants a_n and the integers α_n from the constants c_n. They then develop briefly the notion of a continued fraction "associated" with a general power series (1). A study is made of the Padé approximants and the convergents of (2). This idea leads to the notion of "absolutely regular" power series which include semi-normal power series as a subclass and to which many of the properties of semi-normal series can be extended. Special cases of (2) which represent functions analytic within the unit circle and having this circle as a natural boundary are established by the use of familiar arguments.

W. Leighton (Houston, Tex.).

Agnew, Ralph Palmer. On translations of functions and sets. Bull. Amer. Math. Soc. **46**, 525–530 (1940). [MF 2424]

Let $\{x_n(t)\}$ be a sequence of complex-valued functions measurable over $-\infty<t<\infty$. The author proves the following theorem. In order that the sequence $\{x_n(t)\}$ may be such that, for each real sequence $\{\lambda_n\}$, $\lim_{n\to\infty}x_n(t-\lambda_n)=0$ for almost all t, it is necessary and sufficient that, for each $\delta>0$,

$$\sum_{n=1}^{\infty}\text{l.u.b.}_{-\infty<h<\infty}|E_t\{h\leqq t\leqq h+1; |x_n(t)|\geqq\delta\}|<\infty,$$

where $|E_t\{\cdots\}|$ denotes the (Lebesgue) measure of the set indicated. The author proves two lemmas on translations of sets which he uses in the proof of the theorem.

W. T. Martin (Cambridge, Mass.).

Spain, B. Interpolated derivatives. Proc. Roy. Soc. Edinburgh **60**, 134–140 (1940). [MF 2432]

Essentially the problem of the generalized derivative is a problem in interpolation. The values of the derivatives are known for all integral values of n; for all positive integers, being the ordinary derivatives; for zero, being the function itself; for negative integers, being repeated integrals. Hence if

$$F(r)=f^{(r)}(x), \qquad r=0, 1, 2, \cdots,$$

$$F(r)=\frac{1}{(-r-1)!}\int_a^x(x-u)^{-r-1}f(u)du, \quad r=-1, -2, \cdots,$$

the problem is to find $F(r)$ for non-integral values of r. Of the infinitely many possible functions $F(r)$ two are obtained in the paper.

(1) Starting from the interpolation formula

$$F(n)=\frac{\sin\pi n}{\pi}\sum_{r=-\infty}^{\infty}\frac{(-1)^r F(r)}{n-r},$$

the author derives the expression

$$F(n)=\frac{1}{\Gamma(-n)}\left\{\int_a^x \frac{f(u)du}{(x-u)^{n+1}}+\int_{-\infty}^a du f(u)\int_1^\infty \frac{t^n e^{-t(x-u)}}{\Gamma(n+1)}dt\right\},$$

provided $f(x)$ is an integral function. It is shown that this $F(n)$ can be continued analytically over the complex n-plane. However this definition is unsatisfactory because the operation $D^nD^nf(x)$ cannot be performed.

(2) Starting from another interpolation formula

$$F(n)=\frac{\sin \pi n}{\pi}e^{kn}\sum_{r=-\infty}^{\infty}\frac{(-1)^r F(r)e^{-kr}}{n-r},$$

and ultimately taking the limit as $k\to\infty$, the author obtains the usual formula

$$F(n)=\frac{1}{\Gamma(-n)}\int_a^x \frac{f(u)du}{(x-u)^{n+1}},$$

where $f(x)$ is not necessarily an integral function, and for which the law $D^nD^nf(x)=D^{n+n}f(x)$ holds.

W. E. Milne (Corvallis, Ore.).

Alt, Wilhelm. Über die reellen Funktionen einer reellen Veränderlichen, welche ein rationales Additionstheorem besitzen. Deutsche Math. **5**, 1–12 (1940). [MF 2134]

The problem is that of determining real $f(x)$ and rational $r(x, y)$ such that $f(x)+f(y)=f(r(x, y))$ on the domain of $f(x)$. If $f(x)$, not identically 0 or ∞, satisfies such an equation the $r(x, y)$ is a quotient of polynomials of the form $axy+b(x+y)+c$. For such $r(x, y)$ as admit a differentiable solution $f(x)$, the solution $f(x)=g(x)$, $c\log g(x)$, $c\log|g(x)|$ or $c\arctan g(x)$, where $g(x)=(ax+b)/(cx+d)$. If measurable solutions exist, they are of these forms. If $r(x, y)$ admits any solution, it admits one of the above forms as a solution. By means of Hamel's basis and the known results for the case $r(x, y)=x+y$, the existence of nonmeasurable solutions is shown.

L. W. Cohen (Lexington, Ky.).

Sibagaki, Wasao. Critical points of a real function of n independent variables on a hypersurface defined by $m(<n)$ equations in these variables. Proc. Phys.-Math. Soc. Japan **21**, 535–587 (1939). [MF 1103]

The author studies the critical points of a function $f(x)$ on a surface F in $(x_1\cdots x_n)$-space defined by equations of the form $\phi_\alpha(x)=0$ $(\alpha=1,\cdots,m<n)$. A classification of points of F by means of the rank of the matrix $\|\phi_{\alpha x_i}\|$ is given, together with a geometric interpretation of this classification. The author next defines and obtains elementary properties of normal and abnormal critical points. A classification of nondegenerate critical points is given in terms of the negative index of a quadratic form arising from the second differential of f on F. As indicated by the author, the paper is closely related to one by Bliss on "Normality and abnormality in the calculus of variations" [Trans. Amer. Math. Soc. **43**, 365–376 (1938)].

M. R. Hestenes (Chicago, Ill.).

Cartan, H. et Mandelbrojt, S. Solution du problème d'équivalence des classes de fonctions indéfiniment dérivables. Acta Math. **72**, 31–49 (1940). [MF 2235]

The function $f(x)$, of class C^∞ in an interval I, belongs to the class $\{A_n\}_I$ if to each $x_0\varepsilon I$ there are a neighborhood $V(x_0)$ and a finite $\lambda>0$ such that $|f^{(n)}(x)|\leqq\lambda^n A_n$ $(n=1, 2,\cdots)$ for $x\varepsilon I\cdot V(x_0)$; $\{A_n\}$ is a given sequence of positive numbers (finite or infinite). The authors solve for a finite interval the problem (set by Carleman) of determining necessary and sufficient conditions for two classes $\{A_n\}_I$ and $\{A_n'\}_I$ to be equivalent. The conditions are expressed in terms of two "regularized" sequences $\{A_n^0\}$ and $\{A_n^1\}$ associated with a given $\{A_n\}$. $\{A_n^0\}$ is the "exponential regularization" of $\{A_n\}$ [see also Mandelbrojt, Séries de Fourier et classes quasi-analytiques de fonctions, 1935, pp. 95–96], and $\{A_n^1\}$ can be obtained by setting $B_n=(n^nA_n)^{1/n}$, $(n^nA_n^1)^{1/n}=B_n^0$. Then if I is open a necessary and sufficient condition for $\{A_n\}_I$ to be contained in $\{A_n'\}_I$ is $\limsup_{n=\infty}(A_n^0/A_n')<\infty$; if I is closed or half-open, the condition is the same except that A_n^0 is replaced by A_n^1. The proofs of the sufficiency of the conditions depend on inequalities giving a bound for the kth derivative of a function in terms of bounds for the function itself and its pth derivative $(p>k)$. The authors refer to a forthcoming tract by H. Cartan in the Actualités Scientifiques et Industrielles for these inequalities; the inequalities given by Gorny can also be used [Acta Math. **71**, 317–358 (1939); these Rev. **1**, 137. In the seventh line of the review of this paper, $\binom{n}{k}$ should be replaced by (n/k)]. The authors prove the necessity of the conditions by constructing a series of Tschebycheff polynomials which defines a particular function belonging to $\{A_n\}_I$, having suitable special properties.

R. P. Boas, Jr. (Durham, N. C.).

Mandelbrojt, S. Sur les fonctions indéfiniment dérivables. Acta Math. **72**, 15–29 (1940). [MF 2234]

The author calls a function $f(x)$, of class C^∞ in $a\leqq x\leqq b$, of strongly quasi-analytic type if the series $\sum M_n^{-1/n}$ diverges, where $M_n=\sup_{a\leq x\leq b}|f^{(n)}(x)|$. The class $\{M_n\}_I$ $(I=[a, b]$; notation as in the preceding review) is then quasi-analytic. The author proves that every function of class C^∞ in $[a, b]$ is the sum of two functions of strongly quasi-analytic type in $[a, b]$. The given function is developed in a series of Tschebycheff polynomials; a suitably selected subset of the terms of this series, and the complement of this subset, yield the desired functions of quasi-analytic type. The author's theorem furnishes a strongly negative answer to a problem of Carleman, namely, whether two functions of strongly quasi-analytic type coinciding, together with all their derivatives, at a point are identical [a particular counter-example had previously been given by San Juan, C. R. Congrès Intern. Math. Oslo, 1936, vol. 2, p. 94]. Another application is a sharpening of the Carleman necessary condition for a class of functions to be quasi-analytic.

R. P. Boas, Jr. (Durham, N. C.).

Cartan, Henri. Sur les maxima des dérivées successives d'une fonction. C. R. Acad. Sci. Paris **210**, 431–434 (1940). [MF 1921]

Comme l'avait montré antérieurement l'auteur dans le mémoir analysé plus haut en commun avec S. Mandelbrojt, les limitations $|f^{(n)}(x)|\leqq A_n$, pour tout n $(-1\leqq x\leqq 1)$, entraînent les inégalités

$$|f^{(n)}(0)|\leqq 2e^nA_n^0,\quad |f^{(n)}(x)|\leqq 2[e(1+e^2)]^nA_n',$$

où

$$A_n^0=\overline{\text{borne}}_{r\geq n}\frac{r^n}{S(r)},\quad A_n'=n^{-n}\,\overline{\text{borne}}_{r\geq n}\frac{r^{2n}}{U(r)},$$

avec

$$S(r)=\max_{n\leq r}\frac{r^n}{A_n},\quad U(r)=\max_{n\leq r}\frac{r^{2n}}{n^nA_n}.$$

Les auteurs en déduisaient que toute fonction de la classe $\{A_n\}$, sur un intervalle ouvert (respectivement fermé), appartient à la classe rectifiée A_n^0 (respectivement A_n').

Dans cette note l'auteur montre que ce résultat ne peut pas être amélioré (à la valeur des constantes près); d'une façon précise, pour toute suite A_n, il existe deux fonctions $f(x)$ et $g(x)$ appartenant à la classe $\{A_n\}$ sur l'intervalle $(-1, 1)$ (resp. $[-1, 1]$) et satisfaisant, pour tout n, aux inégalités

$$|f^{(n)}(0)| \geqq A_n^0, \quad |g^{(n)}(1)| \geqq A_n^1.$$

A. Gorny (Clermont-Ferrand).

Gontcharoff, W. Encore sur le théorème de M. Mandelbrojt. Rec. Math. [Mat. Sbornik] N.S. **5** (**47**), 447–450 (1939). (Russian. French summary) [MF 2312]

Rejoinder on Mandelbrojt's review [Zentralblatt Math. **19**, 312–313 (1939)] of a paper by the author [Rec. Math. [Mat. Sbornik] N.S. **3** (**45**), 673–677 (1938)].

Perlin, Irwin E. Indefinitely differentiable functions with prescribed least upper bounds. Bull. Amer. Math. Soc. **46**, 272–273 (1940). [MF 1831]

Let $F(x)$ be indefinitely often differentiable in $0 \leqq x \leqq a$. The author shows that if $\{M_i\}$ is a given sequence for which $M_i a^i > 0$ decreases monotonically then there is a function $F(x)$ such that l.u.b. $|F^{(i)}(x)| = M_i$, $0 \leqq x \leqq a$, $i = 0, 1, 2, \cdots$. *A. C. Schaeffer* (Palo Alto, Calif.).

Kolmogoroff, A. On inequalities between upper bounds of consecutive derivatives of an arbitrary function defined on an infinite interval. Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 3–16 (1939). (Russian. English summary) [MF 2102]

This paper contains the proof of the following theorem announced by the author in C. R. Acad. Sci. Paris **207**, 764–765 (1938). (Theorem I:) Let $f(x)$ be a real function, defined and bounded, together with its first n derivatives, on the real axis; let $M_k(f)$ denote the least upper bound of $|f^{(k)}(x)|$. Then, in order that the triple of positive numbers M_0, M_k, M_n $(0 < k < n)$ should be the quantities $M_0(f)$, $M_k(f)$, $M_n(f)$ for a function $f(x)$, it is necessary and sufficient that

$$M_k^n \leqq C_{n,k} M_0^{n-k} M_n^k, \tag{1}$$

where $C_{n,k} = K_{n-k}/K_n^{(n-k)/n}$,

$$K_n = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}}, \qquad \text{even } n,$$

$$K_n = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{n+1}}, \qquad \text{odd } n.$$

In particular, $1 < C_{n,k} < \pi/2$ for all n and k; $C_{n,n-1} \to \pi/2$ as $n \to \infty$; $C_{n,1} \to 1$ as $n \to \infty$. Thus, as far as the interval $(-\infty, \infty)$ is concerned, the author solves completely a problem initiated (for general n) by Hardy and Littlewood [Proc. London Math. Soc. (2) **11**, 411–478 (1913)]. Hadamard obtained $C_{n,1} \leqq 2^{n-1}$ [C. R. Soc. Math. France **1914**, 68–72]; Gorny obtained $C_{n,k} \leqq 16(2e)^k$ [Acta. Math. **71**, 317–358 (1939); these Rev. **1**, 137]. The case $n = 2$ is a well-known consequence of Taylor's theorem with remainder; the best constant, $C_{2,1} = 2^{1/2}$, appears in a paper by Landau [Proc. London Math. Soc. (2) **13**, 43–49 (1913)], but seems to have been first stated explicitly by Carleman [Les fonctions quasi-analytiques, Uppsala, 1926, p. 12]. The author states that his theorem was proved for $n < 5$ and for $n = 5$, $k = 2$ by G. Shilov. The author points out that it is unnecessary to suppose that $f^{(n)}(x)$ exists everywhere; it is sufficient that $f^{(n-1)}(x)$ should have its Dini derivatives bounded; $M_n(f)$ then denotes their common least upper bound.

The sufficiency of the condition (1) is easily obtained from the fact that equality is attained in (1) for every n and k by the functions

$$f_n(x) = \frac{4}{\pi} \sum_{m=0} \frac{\sin\{(2m+1)x - n\pi/2\}}{(2m+1)^{n+1}}. \tag{2}$$

These functions also play an important part in the proof of the necessity. This proof depends on the following auxiliary proposition, which is sharper than the classical case $n = 2$, $k = 1$ of Theorem I. (Theorem II:) If $\varphi_n(x)$ is a function of the form $af_n(bx+c)$ $(a > 0, b > 0)$ such that $M_0(\varphi_n) = M_0(f)$, $M_n(\varphi_n) = M_n(f)$, and $\varphi_n(x_0) = f(x_0)$, then $|f'(x_0)| \leqq |\varphi_n'(x_0)|$. The proof proceeds by induction. Theorem II is trivial for $n = 1$. If Theorem II is true for $n = m$, the case $k = 1$ of Theorem I is true for $n = m+1$; when this has been established, Theorem II follows for $n = m+1$. Finally, the general case of Theorem I follows easily from the case $k = 1$. The argument is simple and ingenious, depending essentially on the extremal properties of the functions (2). *R. P. Boas, Jr.* (Durham, N. C.).

Soboleff, S. Sur l'évaluation de quelques sommes pour une fonction définie sur un réseau. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 5–16 (1940). (Russian. French summary) [MF 1978]

Let D be an n-dimensional region (there is some restriction) containing an n-dimensional cubical lattice of mesh h, and let $u_{\alpha_1\alpha_2\cdots\alpha_n} = u(\alpha_1 h, \alpha_2 h, \cdots, \alpha_n h)$ be a function defined on the vertices of the lattice. The following inequality is proved: $\sigma_0 \leqq M\sigma_l + L\sigma$, where

$$\sigma_l = \sum_{\beta_1+\cdots+\beta_n=l} \Big[\sum_{D'} |\Delta_{\beta_1\beta_2\cdots\beta_n} u|^p h^n\Big]^{1/p},$$

$$\sigma_0 = \Big[\sum_{D'} |u_{\alpha_1\alpha_2\cdots\alpha_n}|^q h^n\Big]^{1/q},$$

$$\sigma = \sum_{D'} |u_{\alpha_1\alpha_2\cdots\alpha_n}| h^n;$$

D' is a connected subset of D depending on l; $p > 1$, $q > 1$, $1/q = (1/p) - (1/n)$; and M and L depend only upon the form of D, not upon the choice of u or h. The author has previously proved an analogous inequality for integrals [Rec. Math. [Mat. Sbornik] N.S. **4** (**46**), 471–497 (1939)]. *J. V. Wehausen* (New York, N. Y.).

Pizzetti, Ernesto. Osservazioni sulle medie esponenziali e baso-esponenziali. Metron **13**, no. 4, 3–15 (1939). [MF 2533]

Let $1 < x_1 < x_2 < \cdots < x_n$ be n given numbers; the author defines five different mean values y, z, ω, α and β by $ny^y = \sum x_i^{x_i}$, $nc^z = \sum c^{x_i}$, $n\omega^\omega = \sum x_i^{x_i}$, $n\alpha^{1/\alpha} = \sum x_i^{1/x_i}$ and $nc^{1/\beta} = \sum c^{1/x_i}$, where c and d are given positive numbers. Some relations between these quantities and some special cases are considered. *W. Feller* (Providence, R. I.).

Calculus

***Tricomi, Francesco. Lezioni di analisi matematica.** Parte seconda. 4th ed. Cedam, Padova, 1939. viii+355 pp. L. 85.

This is the second volume of a text-book on advanced calculus covering the following topics: (1) The definite integral;

upper and lower Riemann integrals, mean value theorems, extensions. (2) Integration in finite terms; rational, algebraic functions. (3) Series expansions and numerical calculations; uniform convergence, power series, Taylor's formula and series, elementary analytic functions, Fourier series, interpolation and numerical integration. (4) Functions of several variables; derivatives, homogeneous functions, total differentials, Taylor's formula and series, implicit functions, jacobians, functional dependence, maxima and minima, least squares, Schwarz's inequality. (5) Elements of differential geometry; vectors, space curves including length, curvature, torsion, Frenet's formulas, singularities of plane curves, envelopes, surfaces including first fundamental form and curvature. (6) Integration of functions of several variables; geometrical applications, formula of Gauss. (7) Ordinary differential equations; successive approximations, singular solutions, linear equations, integration by series. (8) On partial differential equations and the calculus of variations; elements, equation of Laplace, Green's theorem, vibrating string, fundamental problem of calculus of variations, the brachystochrone and the isoperimetric problem. The presentation is clear and simple but appears to be rigorous. There are no exercises but good references for supplementary reading. *E. Hille* (New Haven, Conn.).

***Dresden, Arnold. Introduction to the Calculus.** Henry Holt and Co., New York, 1940. xii+428 pp. $3.40.

An undergraduate text emphasizing formal logical aspects. Opening chapters deal with "Sets of points, limits," and "Continuous functions," respectively. There is a chapter on vectorial applications to mechanics.

A. A. Bennett (Providence, R. I.).

***Stewart, C. A. Advanced Calculus.** Methuen & Co., Ltd., London, 1940. xviii+523 pp. 25s.

This book provides a very thorough course in the differential and integral calculus. It includes all of the topics discussed in most American textbooks on advanced calculus but treats them in a fuller and more fundamental way than is usually the case in such books. Among the subjects which are rarely treated in such books but which do appear in the book under review should be mentioned the elements of the theory of sets, the concept of the Lebesgue integral, the elements of tensor analysis, asymptotic expansion and Bernoulli polynomials. There are very extensive lists of problems at the end of each of the twelve chapters.

A. Dresden (Swarthmore, Pa.).

***Middlemiss, Ross R. Differential and Integral Calculus.** McGraw-Hill Book Company, Inc., New York, 1940. x+416 pp. $2.50.

Bohr, Harald. On the logarithmic and exponential functions. Mat. Tidsskr. A. **1939**, 46–81 (1939). (Danish) [MF 1058]

Two lectures on the different methods of introducing the logarithmic and the exponential functions. The survey is in the first part historical, in the second systematic.

W. Feller (Providence, R. I.).

Gonzalez, Mario O. A generalization of Cauchy's formula and its relation to the development in Burmann series. Revista Ci., Lima **41**, 549–553 (1939). (Spanish) [MF 1646]

It is shown that the procedure allowing the derivation of the Taylor series from the mean value theorem of differential calculus may also be applied to derive the Burmann series from Cauchy's formula

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\frac{f'(\xi)}{\varphi'(\xi)}, \qquad a<\xi<b.$$

I. J. Schoenberg (Waterville, Me.).

Rashevski, P. On a general theorem concerning the contact of curves (a generalization of the mean value theorem of the differential calculus). Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 185–193 (1939). (Russian. English summary) [MF 2114]

Let the curves C, $y=f(x)$, and Γ, $y=g(x)$, be any two curves situated within some fixed rectangle $a_1\leqq x\leqq a_2$, $b_1\leqq y\leqq b_2$, and having $n+1$ points in common. They are to have as many derivatives as necessary. In addition, let n differential equations be given, $y^{(s)}=f_s(x, y, y', \cdots, y^{(s-1)})$, $s=1, \cdots, n$, where the integral curves Γ_s of the sth equation are contained in those of the $(s+1)$th equation and where the curve Γ is contained in Γ_1. The functions f_s are defined for x and y in the fixed rectangle and for $m_s\leqq y^{(s)}\leqq M_s$ where, however, all the values of $y', \cdots, y^{(s-1)}$ along the curve C must be contained in this region. The theorem then asserts that among the integral curves of Γ_s there is at least one which has a point of contact of order n with the curve C.

J. V. Wehausen (New York, N. Y.).

Ionesco, D. V. L'application d'une formule de T. J. Stieltjes à un problème de M. D. Pompeiu. Acad. Roum. Bull. Sect. Sci. **21**, 214–218 (1939). [MF 1509]

The author considers a formula of Stieltjes,

$$(1)\quad \lambda(x)f(x)+\lambda(x_1)f(x_1)+\cdots+\lambda(x_n)f(x_n)=\frac{1}{n!}f^{(n)}(\xi);$$

here $\lambda(x)=1/\psi'(x)$ with $\psi(z)=(z-x)(z-x_1)\cdots(z-x_n)$; $x, x_1, \cdots, x_n$ are on the interval (a, b); ξ lies between the least and the greatest of the numbers $x, x_1, \cdots, x_n$; f is n times derivable, with $f^{(n)}$ finite on (a, b). It is proved that, if in (1) f is any polynomial of degree $n+1$, one has (2) $(n+1)\xi=x_1+\cdots+x_n$ and that, in fact, (2) is characteristic for polynomials of degree $n+1$.

W. J. Trjitzinsky (Urbana, Ill.).

Ionesco, D. V. Généralisation d'une équation fonctionnelle rencontrée par G. Darboux. Acad. Roum. Bull. Sect. Sci. **22**, 165–167 (1939). [MF 1504]

Ionesco, D. V. Applications d'une formule généralisée de G. Darboux. Acad. Roum. Bull. Sect. Sci. **22**, 168–171 (1939). [MF 1505]

Darboux remarked that for polynomials $\varphi(x)$ of degree not greater than three the following formula holds:

$$\int_0^h \varphi(x)dx=\frac{h}{6}\{\varphi(0)+4\varphi(\tfrac{1}{2}h)+\varphi(h)\}.$$

In the first paper the author notes that for polynomials of degree not greater than two the following more general formula holds:

$$\int_\lambda^\mu \varphi(x)dx=\frac{\mu-\lambda}{6}\left\{\varphi(\lambda)+4\varphi\left(\frac{\lambda+\mu}{2}\right)+\varphi(\mu)\right\}.$$

He also gives generalizations to polynomials of any degree, even or odd. The second paper gives geometric applications.

O. Szász (Cincinnati, Ohio).

Rinehart, R. F. On extrema of functions which satisfy certain symmetry conditions. Amer. Math. Monthly **47**, 145–152 (1940). [MF 1733]

The following definitions are needed to state the theorem proved in this paper. A region R in euclidean n-space $(x_1, x_2, \cdots, x_n)$ is called symmetric with respect to $x_1, \cdots, x_q$, $q \leqq n$, if, whenever a point $(a_1, \cdots, a_n)$ is in R, so is also every point obtained by a permutation of $a_1, \cdots, a_q$. A function $S(x_1, \cdots, x_n)$, given in such a region R, is symmetric with respect to $x_1, \cdots, x_q$ if it is invariant under all permutations of $x_1, \cdots, x_q$. Let there be given functions S, $V^{(i)}$, $i=1, \cdots, k<n$ in a region R, all symmetric with respect to the variables $x_1, \cdots, x_{n-k+1}$, and possessing continuous partial derivatives of the first two orders with respect to all the variables $x_1, \cdots, x_n$. Let $P=(a, \cdots, a, a_{n-k+2}, \cdots, a_n)$ be a point in R which satisfies the equations $V^{(i)}=0$. Let the Jacobian J of the functions $V^{(i)}$ with respect to $x_{n-k+1}, \cdots, x_n$ be different from zero at P. Using subscripts to refer to partial differentiation, put

$$\Delta=\frac{1}{J}\begin{vmatrix} S_{11}-S_{12} & S_{n-k+1} & \cdots & S_n \\ V_{11}^{(1)}-V_{12}^{(1)} & V_{n-k+1}^{(1)} & \cdots & V_n^{(1)} \\ \vdots & \vdots & & \vdots \\ V_{11}^{(k)}-V_{12}^{(k)} & V_{n-k+1}^{(k)} & \cdots & V_n^{(k)} \end{vmatrix}.$$

Then the function S, where the variables are subject to the side conditions $V^{(i)}=0$, has a relative extremum at P provided $\Delta \neq 0$ at P. More precisely, the extremum is a maximum or a minimum according as $\Delta<0$ or $\Delta>0$ at P. The author observes that the set of the points P described above is independent of the function S. Many interesting special cases come under the above general theorem. The proof consists of an application of the standard elementary methods.

T. Radó (Columbus, Ohio).

Capelli, Pedro F. On holomorphic and polygonal functions of a binary complex variable. An. Soc. Ci. Argentina **128**, 154–174 (1939). (Spanish) [MF 1080]

The author considers complex numbers, the imaginary part α of which is a root of the equation $\alpha^2=\mu+\nu\alpha$, with μ, ν real. Some fundamental properties of such numbers and of their functions are stated. *W. Feller.*

Tricomi, Francesco. Su di un integrale doppio presentatosi in aerodinamica. Atti Accad. Sci. Torino **75**, 17–25 (1939). [MF 1882]

It is proved that for $\omega \geqq 0$

$$\int_0^\infty \cos \omega x dx \int_{-1}^{1}\left(1-\frac{x}{(x^2+y^2)^{\frac{1}{2}}}\right)\frac{dy}{(1-y^2)^{\frac{1}{2}}} = 2\int_0^{\pi/2}\int_0^{\pi/2} e^{-\omega \sin\phi \sin\psi} \sin\phi \sin\psi d\phi d\psi.$$

Using the Laplace transform, it follows that the asymptotic behavior of the integral as $\omega \to \infty$ is described by $(\log \omega)/\omega^2$.

W. Feller (Providence, R. I.).

Ferrar, W. L. Ratio tests for the convergence of integrals. Edinburgh Math. Notes **1939**, no. 31, i–iii (1939). [MF 992]

The author translates the usual ratio tests for infinite series into the corresponding tests for infinite integrals. As he remarks himself, these integral tests, while showing the underlying ideas in a particularly simple way, are quite useless, and he explains why this is so. *W. Rogosinski.*

Puig Adam, P. Simplified demonstration of the De Moivre-Stirling formula and a geometrical limit for the error. Revista Mat. Hisp.-Amer. (3) **1**, 21–26 (1939). (Spanish) [MF 1670]

Stirling's formula in the form

$$n!=n^n e^{-n}(2\pi n)^{\frac{1}{2}} e^{\theta/(12n)}, \qquad 0<\theta<1,$$

is proved by approximating geometrically the area under the curve $y=\log x$. A generalization to other curves is indicated. *R. P. Boas, Jr.* (Durham, N. C.).

Huntington, E. V. Stirling's formula with remainder. Biometrika **31**, 390 (1940). [MF 2271]

This note carries the expansion of the factor ε^p, which occurs in Stirling's formula, forward so as to include the term in n^{-9}; it gives bounds, in terms of n, of the remainder after the term in n^{-k}, for $k=0, 1, \cdots, 8$. *A. Dresden.*

Tzénoff, Iv. Points simples et points singuliers des courbes planes. Ann. Univ. Sofia. II. Fac. Phys. Math. Livre 1. **35**, 251–356 (1939). (Bulgarian. French summary) [MF 1377]

Tzénoff, Iv. Points simples et points singuliers des courbes planes. Enseignement Math. **38**, 92–116 (1940). [MF 2252]

Discussion of regular and singular points of plane curves using a vector notation. *J. D. Tamarkin.*

Dolaptschiew, Bl. Über eine Art von Zylinderkurven. Ann. Univ. Sofia. II. Fac. Phys. Math. Livre 1. **35**, 357–363 (1939). (Bulgarian. German summary) [MF 1378]

Proof that the curve $x=R\cos v$, $y=R\sin v$, $z=a(e^v+e^{-v})$ gives a figure of equilibrium of a flexible, inextensible, homogeneous string which it assumes on the surface of a cylinder under action of the force of gravity. *J. D. Tamarkin.*

Ruban, A. N. Sur le problème du cylindre flottant. C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 350–352 (1939). [MF 2071]

Salgaller, V. et Kostelianetz, P. Sur le problème du cylindre flottant. C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 353–355 (1939). [MF 2072]

Discussion of curves, for which chords of fixed length cut off constant areas. *F. John* (Lexington, Ky.).

Opatowski, I. Force lines in Newtonian bidimensional fields. Some applications and generalizations. Revista Ci., Lima **41**, 485–502 (1939). [MF 1643]

The author studies the lines of force of Newtonian potentials which are independent of ϕ when expressed in helicoidal coordinates (r, ϕ, w) or spiral coordinates (ϕ, θ, l); here r, ϕ, θ stand for ordinary polar coordinates, $w=z-m\phi$, $l=re^{-m\phi}$, $m=$const. *W. Feller* (Providence, R. I.).

Berzolari, Luigi. Sull'equazione differenziale di un sistema ∞^1 di coniche osculatrici a una conica data. Boll. Un. Mat. Ital. (2) **2**, 1–10 (1939). [MF 1305]

Costruita l'equazione differenziale del primo ordine, che ha per curve integrali le coniche osculatrici a una data conica C e passanti per due dati punti P, Q, uno situato su C e l'altro in posizione generica rispetto a C, si determina il luogo dei punti in cui si toccano due coniche distinte del detto sistema (tac-locus, secondo una denominazione di A. Cayley), studiando alcune particolorità di esso.

G. Cimmino (Bologna).

Theory of Sets, Theory of Functions of Real Variables

Maximoff, Isaie. Sur la séparabilité d'ensembles. Acad. Roum. Bull. Sect. Sci. **22**, 384–389 (1940). [MF 2600]

Let the family of a-sets be closed under countable intersection and countable union. Call a set which is both an a-set and the complement of an a-set a b-set. Call the intersection of $\aleph_1$ a-sets an aa-set. Call the intersection of $\aleph_1$ b-sets a bb-set. The author proves that disjoint aa-sets can be separated by disjoint bb-sets and that if E_1 and E_2 are aa-sets then E_1-E_2 and E_2-E_1 can be separated by disjoint complements of aa-sets. The proof is carried out for the special case that the family of a-sets is the family of projective sets of class n in the space of irrational numbers. No use is made of special properties of these families. The paper is not free from typographical errors. The last remark cannot be proved by the same line of reasoning.

J. W. Tukey (Princeton, N. J.).

Maximoff, Isaie. Sur le système de Souslin d'ensembles dans l'espace transfini. Bull. Amer. Math. Soc. **46**, 543–550 (1940). [MF 2428]

The author generalizes the usual construction of the families of Borel and analytic sets over a given family of sets by considering unions of $\aleph_i$ sets, intersections of $\aleph_i$ sets and a generalization of the analytic operation, whose result is the union of the sets $E(\nu)=F(n_1)F(n_1n_2)\cdots F(n_1n_2\cdots n_k)\cdots$ $(k<\omega_0)$, where the n's run through all ordinals less than ω_{i+1}. We refer to sets obtained in this way as "Borel" and "analytic" sets. It is asserted that any "Borel" set is "analytic," but this may easily be disproved. It is proved that: (1) Any set "analytic" over the family of "analytic" sets is itself "analytic." (2) The union of $\aleph_i$ "analytic" sets is an "analytic" set. (3) The intersection of $\aleph_0$ "analytic" sets is "analytic." (4) Each "analytic" set (each complement of an "analytic" set) is the union of $\aleph_{i+1}$ constituents. Under the additional hypothesis that the difference of two Borel sets is a Borel set [this hypothesis is omitted] it follows that these constituents are Borel sets. No use is made of the fact that the sets in question lie in a special space, called "transfinite space of order i," and the special argument used to transfer the last theorem to the ordinary Baire space is unnecessary. *J. W. Tukey.*

Inagaki, Takeshi. Le problème de Souslin dans les espaces abstraits. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **8**, 25–46 (1939). [MF 2183]

The author develops certain theorems for accessible space which permit the formulation of Souslin's problem in terms of the kernel (largest dense-in-itself subset) of a set. Let S' be an accessible space and γ an ordinal. A well-ordered family $\{G_\alpha\}$, $0\leqq\alpha<\gamma$, of strictly decreasing open sets in S' is called a decreasing sequence of type γ. In S' if every decreasing sequence is of type at most γ_0, γ_0 an ordinal of the third class, then every family of disjoint open sets is at most denumerable. If S_1 is an accessible neighborhood space satisfying the first denumerability axiom, then every decreasing sequence is of type at most Ω if and only if every nondenumerable set is separable or dense in S_1. In S_1 the property that every decreasing sequence is of type at most Ω is equivalent to the property that the intersection of a decreasing sequence of type Ω is a frontier set.

A space S' is 0-separable if every nondenumerable family $F=\{E\}$ of non-separable disjoint sets contains a denumerable family F^* such that if $E_0\varepsilon F-F^*$ and $G\supset E_0$, G open, then G has a point in some $E\varepsilon F^*$. In S' the property of 0-separability is equivalent to the property that every decreasing sequence is of type less than Ω^2. Let S^* be ordered, dense, without first or last point and Dedekindean. In S^* the existence of a nondenumerable family of disjoint intervals is equivalent to the existence of a decreasing sequence of type at least Ω^2. Let S be a Souslin space, that is, one with the properties of S^* and such that every family of disjoint intervals is at most denumerable. The separability of S is equivalent to the statement that if a nondenumerable E is separable or has a non-empty kernel, then a nondenumerable E is separable or has a kernel dense in S.

L. W. Cohen (Lexington, Ky.).

Inagaki, Takeshi. Les espaces abstraits et les ensembles ordonnés. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **8**, 145–162 (1940). [MF 2188]

In a neighborhood space R let p_1R be the lower bound of the powers of sets dense in R and let p_2R be the upper bound of powers of families of disjoint neighborhoods in R. The following is an example in which $p_1R>p_2R$: the points are real numbers $x_0, x_1, \cdots, x_\alpha, \cdots \mid \Omega$; the neighborhoods $V^*(x_\alpha)$ are the sets of $x_\beta\varepsilon R$ such that $\alpha\leqq\beta<\Omega$ and $x_\beta\varepsilon V(x_\alpha)$, $V(x_\alpha)$ an open interval containing x_α. This space is hypercondensed, 0-separable [cf. the preceding review] and has no separable nondenumerable subset. Examples are given showing that the properties "0-separable" and "hypercondensed" are independent. It is shown that in a nonseparable Souslin space the class of separable sets is the class of nondense sets and that a linearly ordered, Dedekindean space in which the classes of separable sets and of nondense sets coincide is a non-separable Souslin space. A linearly ordered, Dedekindean space in which every separable set is nondense is exhibited. The paper concludes with the theorem that a linearly ordered, Dedekindean space R in which every nondense set is separable is similar to the space of real numbers if and only if there is a neighborhood system $V_n(p)$ for R in which $V_n(p)\supset V_{n+1}(p)$ and $\prod_k V_{n_k}(p_k)$, $n_k\geqq k$, contains at most one point. *L. W. Cohen.*

Kunugui, Kinjiro. Contribution à la théorie des ensembles boreliens et analytiques, II. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **8**, 1–24 (1939). [MF 2182]

The author here continues his studies of certain operations performed on Borel, analytic and projective sets. [For the first article see the same journal **7**, 161–189 (1939)]. If M is any plane set, the set $\Delta(M)$ is the set of points $(a, 0)$ such that the vertical line $x=a$ intersects M in an F_σ (non-null). If M is a Borel set specialized in a certain way, then $\Delta(M)$ is the complement of an analytic set. The proof of this theorem is difficult and occupies about half the paper. The remaining pages are devoted to a study of the atomic elements of sieves, and it is shown that for any denumerable type of order the atomic set is a Borel set.

D. Montgomery (Northampton, Mass.).

Kunugui, Kinjiro. Contribution à la théorie des ensembles boreliens et analytiques, III. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **8**, 79–108 (1940). [MF 2186]

[Cf. the preceding review.] This section of the work begins with a study of the δs-operations of Hausdorff. Results are obtained which are useful in the study of sections carried out in the latter part of the paper. If M is any plane set, the set $\Gamma(M)$ is the totality of points $(a, 0)$ such that the line $x=a$ intersects M in a non-null closed set. It is shown

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that, if M is any Borel set, then $\Gamma(M)$ is the complement of an analytic set. It is a corollary of this theorem that if every vertical section of a Borel set is closed then the projection of the set is a Borel set. There are several other corollaries relating the results above to topological products of various types of spaces. There are also applications to implicit functions. *D. Montgomery* (Northampton, Mass.).

Kunugui, Kinjiro. Sur un problème de M. E. Szpilrajn. Proc. Imp. Acad. Tokyo **16**, 73–78 (1940). [MF 2217]

Let M be any set situated in the plane and let $\Delta(M)$ be the set of all points x_0 on the x-axis such that the vertical line $x=x_0$ intersects M in a non-null F_σ. It is shown that if M is a Borel set, then $\Delta(M)$ is the complement of an analytic set. As an immediate corollary it follows that, if M is a Borel set all of whose vertical sections are F_σ sets, then the projection of M is a Borel set. *D. Montgomery.*

Novikoff, P. C. On projections of some B-sets. Doklady Akad. Nauk SSSR (N.S.) **23**, 863–864 (1939). (Russian) [MF 1963]

If A is a set in the plane, denote by $p(A)$ the set of x_0's such that A intersects the line $x=x_0$ in a closed set. It is proved that, if A is a Borel set, then $p(A)$ is a CA-set. This extends a result of Kunugui [Proc. Imp. Acad. Tokyo **13**, 287–291 (1937)], who showed that, if A is a G_δ, then $p(A)$ is a CA-set. *J. W. Tukey* (Princeton, N. J.).

Liapounoff, A. Sur une propriété des δs-opérations. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1939**, 407–412 (1939). (Russian. French summary) [MF 2087]

An A_n'-set is a projection of a uniform CA_{n-1}-set. A set which is both a A_n'-set and a CA_n'-set is a B_n'-set. The author proves that the class of B_n'-sets is invariant under every δs-operation whose reduced complete base [L. Kantorovitch and E. Livenson, Fund. Math. **18**, 214–279 (1932)] is a B_n'-set. It follows that the smallest family of sets including both A_2-sets and CA_2-sets and invariant under complementation and the operation (A) is contained in the class B_3'. Similar methods prove that the result of a δs-operation whose reduced complete base is a CA-set on Borel sets is a CA-set. *J. W. Tukey* (Princeton, N. J.).

Liapounoff, A. Séparabilité multiple pour le cas de l'opération (A). Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1939**, 539–552 (1939). (Russian. French summary) [MF 2095]

Let M be a field of sets; $B(M)$ be the Borel ring over M; $A(M)$ be the class of sets analytic over M; and $CA(M)$ be the class of complements of sets of $A(M)$. Let $A(\{W_{n_1n_2\cdots n_k}\})$ be the result of the analytic operation on the system $\{W_{n_1n_2\cdots n_k}\}$. The author proves that: (1) If each set of $\{E_{n_1n_2\cdots n_k}\}$ belongs to $A(M)$ and if $A(\{E_{n_1n_2\cdots n_k}\})=0$, then there exist sets $H_{n_1n_2\cdots n_k}$ of $B(M)$ such that $H_{n_1n_2\cdots n_k}\supset E_{n_1n_2\cdots n_k}$ and $A(\{H_{n_1n_2\cdots n_k}\})=0$. He also proves that: (2) If each set of $\{E_{n_1n_2\cdots n_k}\}$ belongs to $A(M)$, then there exist sets $H_{n_1n_2\cdots n_k}$ of $CA(M)$ such that $H_{n_1n_2\cdots n_k}\supset E_{n_1n_2\cdots n_k}-A(\{E_{n_1n_2\cdots n_k}\})$ and $A(\{H_{n_1n_2\cdots n_k}\})=0$. The author also proves some results about the non-separability of systems of $CA(M)$ sets in a Baire space. *J. W. Tukey.*

Kondô, Motokiti. Sur la représentation paramétrique des ensembles. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **8**, 173–220 (1940). [MF 2190]

The author determines the family of continuous (one-one and continuous) images of a space N in a complete separable metric space R in several cases. If N is an absolute F_σ then K_0 is the set at which N is locally uncountable and K_1 is the subset of K_0 where K_0 is not locally G_δ. Absolute F_σ are classified into six types in accordance which the existence and properties of these sets. By the type of a set we include the knowledge of whether or not it is an absolute F_σ, its type in the sense just mentioned, and in addition, if it is scattered (clairsemé), the index and number of points in its last non-vanishing derivative. I is the space of irrational numbers, and C is the complement of the analytic set of Lebesgue. When N is an analytic subset of I, the family of its continuous images depends only on its countability, compactness and type. The possible families are determined. When $N=C$, the family of its continuous images consists of all non-empty projective sets of class 2. When N is an uncountable Borel subset of I, the family of its one-one continuous images depends on the type of N. The families are those of all sets whose symmetric difference from a set of a given family is countable. If $E\subset R$ is projective of class 2, then there is a set E^* which is a continuous image of C, such that both $E-E^*$ and E^*-E are Borel sets. The paper concludes with results about the existence of sets whose continuous images have a definite relation to a given family of sets of the power of the continuum. There are examples of connected plane Borel sets of preassigned class (>1) all of whose one-one continuous images in the plane are of the same Borel class and of connected plane Borel sets of preassigned class (>1) whose one-one continuous images include sets of any other preassigned Borel class. *J. W. Tukey* (Princeton, N. J.).

Best, E. A closed dimensionless linear set. Proc. Edinburgh Math. Soc. (2) **6**, 105–108 (1939). [MF 1523]

The author constructs a closed linear set E which is dimensionless in the sense of Hausdorff's measure theory [Math. Ann. **79**, 157–159 (1918)]; this means that given any function $\lambda(t)$ of the real variable t, defined in some interval $(0\leqq t\leqq t_0)$ in which it is continuous, concave and strictly increasing, the Hausdorff measure of E with respect to the function λ is either 0 or ∞. The set E is constructed as follows: Let E_r be the Cantor ternary set defined in the interval $(1/r+1, 1/r)$, then the set $E=\sum_{r=1}^{\infty}E_r$ has the desired property. The author remarks that the existence of closed dimensionless sets in the plane is trivial (for example, any set which is the sum of a countable number of disjoint congruent segments has this property). *W. Hurewicz.*

Best, E. On sets of fractional dimensions. Proc. Cambridge Philos. Soc. **36**, 152–159 (1940). [MF 1716]

The author considers the set of numbers x that can be expressed in the form

$$x=\frac{x_1}{1!}+\frac{x_2}{2!}+\frac{x_3}{3!}+\cdots, \quad x_r<r, \text{ a positive integer},$$

with bounded x_r's. He shows that this set has the dimension function $h(t)=b^{-u}$, where $t=e^u u^{-u-\frac{1}{2}}(b-1)(2\pi)^{-\frac{1}{2}}$ in the sense of Hausdorff's measure theory [Math. Ann. **79**, 157–159 (1918)]. *W. Hurewicz* (Chapel Hill, N. C.).

Kronrod, A. Sur la structure de l'ensemble des points de discontinuité d'une fonction dérivable en ses points de continuité. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1939**, 569–578 (1939). (Russian. French summary) [MF 2097]

The following theorem is proved. In order that a set of points be a set of points of discontinuity for some function

having a finite derivative at every point of continuity it is necessary and sufficient that the set be simultaneously of types F_σ and G_δ. The half of the theorem concerning sufficiency is generalized by imposing less stringent conditions upon the function (a generalized type of Lipschitz condition is substituted for differentiability). It is also proved that, if the set of points of discontinuity is not of type G_δ, the function has no derivative in continuum-many points of continuity. *J. V. Wehausen* (New York, N. Y.).

Minakshisundaram, S. On the roots of a continuous non-differentiable function. J. Indian Math. Soc. (N.S.) **4**, 31–33 (1940). [MF 2032]

Let $f(x)$ be continuous and non-differentiable on the interval $(0, 1)$ with lower and upper bounds l and u. For $l \leqq \alpha \leqq u$, let $S(\alpha)$ be the points on $(0, 1)$ for which $f(x) = \alpha$. Divide the interval (l, u) into the sets A: $S(\alpha)$ is of positive measure; B: $S(\alpha)$ is non-enumerable but of zero measure; C: $S(\alpha)$ is enumerable. It is shown that A is at most enumerable; B has the measure of the interval (l, u); C is at most of zero measure. In other words, a continuous non-differentiable function takes almost every value more than enumerably often. For the Weierstrass non-differentiable function $W(x) = \sum a^n \cos b^n \pi x$ the set C is not empty and is at most enumerable. Except possibly for an enumerable set α which are proper maxima and minima of $W(x)$, the set $S(\alpha)$ is perfect, and, except for a set α of zero measure, the set $S(\alpha)$ is non-dense and perfect. *R. L. Jeffery.*

Broudno, A. Sur les fonctions uniformement continues sur des ensembles mesurables *B*. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 105–112 (1940). (Russian. French summary) [MF 1983]

Let $f(x) = \Gamma \lim_{n \to \infty} f_n(x)$ on the closed interval $[0, 1]$ if for each $x \in [0, 1]$ there exists $n(x)$ such that $f_n(x) = f(x)$ for $n \geqq n(x)$. By defining as functions of class zero all functions continuous on $[0, 1]$ and by applying transfinitely the operation $\Gamma \lim_{n \to \infty}$, one obtains a classification of functions similar to the Baire classification. This classification, due to Glivenko, is related to Borel's classification of sets as follows. Denote by N^1 and M^1, respectively, the classes of closed and open sets, and by N^α and M^α the classes of sets which are the intersection and union, respectively, of sets belonging to classes N^β or M^β for $\beta < \alpha$. Then $f(x)$ belongs to the class α of Glivenko if and only if there is a decomposition $[0, 1] = \sum_{n=1}^{\infty} N_n^\alpha$ such that $f(x)$ is uniformly continuous on each N_n^α. The relationship between this classification, the classification of Baire and that of Moore-Chittenden [cf. Chittenden, Trans. Amer. Math. Soc. **23**, 1–15 (1922)] is also discussed. *J. V. Wehausen.*

Torrance, Esther McCormick. Superposition on monotonic functions. Duke Math. J. **6**, 307–317 (1940). [MF 2316]

The author studies the properties of $h(x) = f[g(x)]$ and $k(x) = f[g^{-1}(x)]$, where all functions are defined on $I = [0, 1]$ and assume only values in this range; g is monotonic. If H denotes the subset of I on which g is continuous and one-to-one, $H' = g(H)$, then the two G_δ's H and H' are homeomorphic via g. Several properties of sets are shown to be invariant under g or g^{-1}, including the property of belonging to a given Borel class of order $\alpha (\alpha > 1)$, the restricted Baire property, perfect measurability, and others. If $f(x)$ is a Baire function of class α $(\alpha > 1)$ then $f[g(x)]$ is shown to belong to the same class. The question of invariance of further set properties is investigated, both positive and negative results being obtained, and the situation contrasted with the one which arises from assuming only that g is continuous. Measurability questions are considered, the main theorems being (a) for h to be measurable for every measurable f it is necessary and sufficient that g^{-1} be absolutely continuous on H', and (b) for h to be measurable for every f it is necessary and sufficient that $mH = 0$. Analogous results are obtained when measurability is replaced by the Baire property. The theorems on measure are restated as theorems concerning Lebesgue-Stieltjes integrals. [On p. 317, assertion (3), H should be replaced by H'; an equivalent statement is that the only Lebesgue-Stieltjes measures with respect to which all sets are measurable are those in which all the measure is concentrated on a denumerable set of points.] *J. A. Clarkson.*

Mulholland, H. P. On the total variation of a function of two variables. Proc. London Math. Soc. (2) **46**, 290–311 (1940). [MF 2361]

Consider a function $f(x, y)$ continuous in a closed convex region R; denote by R^θ the region obtained by rotating R about the origin through the angle Θ, and put $f^\theta(x, y) = f(x \cos \Theta + y \sin \Theta, \ -x \sin \Theta + y \cos \Theta)$, where $(x, y) \in R^\theta$. Finally put

$$T_x[y; f^\theta; R^\theta] = \int_{(t, y) \in R^\theta} |d_t f^\theta(t, y)|,$$

and

$$W(\Theta) = \int T_x[y; f^\theta; R^\theta] dy.$$

With this notation, $f(x, y)$ is of bounded variation in Tonelli's sense if $W(0)$ and $W(\pi/2)$ both exist and are finite. The author shows that this is the case if, and only if, $V_R[f] = \frac{1}{2} \int_0^\pi W(\Theta) d\Theta$ is finite; thus, being independent of the choice of the axis, $V_R[f]$ appears to be a natural definition of the total variation of $f(x, y)$ over R. With this definition the author also obtains generalizations of two important formulae for functions of one variable: namely of Banach's formula $\int_a^b |dg(x)| = \int N(y) dy$, where $N(y)$ is the number of values x for which $g(x) = y$, and of the formula

$$\int_a^b |dg(x)| = \lim_{h \to +0} \int_a^{b-h} \left| \frac{f(x+h) - f(x)}{h} \right| dx.$$

W. Feller (Providence, R. I.).

Tonelli, Leonida. Sulle funzioni di intervallo. Ann. Scuola Norm. Super. Pisa (2) **8**, 309–321 (1939). [MF 2044]

Let $\Phi(\delta)$ be an interval function, defined for all subintervals δ of an interval (a, b), which satisfies the following two conditions: (a) for each $\epsilon > 0$, there exists a $\lambda_\epsilon > 0$ such that, if δ is an interval of length not greater than λ_ϵ, then

$$\Phi(\delta) \leqq \Phi(\delta_1) + \cdots + \Phi(\delta_n) + \epsilon \cdot |\delta|, \qquad |\delta| = \text{length of } \delta,$$

for every subdivision of δ into intervals $\delta_1, \cdots, \delta_n$; (b) for each $\sigma > 0$, there exists a $\lambda_\sigma > 0$ such that

$$\Phi(\delta_1) + \Phi(\delta_2) < \Phi(\delta) + \sigma$$

for each interval δ of length not greater than λ_σ, which is divided into two intervals δ_1 and δ_2.

Concerning such set functions, the author proves that either the Burkill integral of Φ exists on (a, b) or the lower Burkill integral equals $+\infty$; in other words, the Burkill integral always exists in the extended sense, admitting $+\infty$ as a possible value (not $-\infty$). Using this result, the author

shows first that, if $f(x)$ has only discontinuities of the first kind with $f(a)=f(a^+)$, $f(b)=f(b^-)$, and $f(x^-)\leqq f(x)\leqq f(x^+)$, and if $\Phi(\delta)=|f(\beta)-f(\alpha)|$, $\delta=(\alpha,\beta)$, then the Burkill integral of $\Phi(\delta)$ equals the variation of $f(x)$ on (a,b). Let $F(x,y,x',y')$ be any function which is continuous in its arguments for (x,y) in a bounded domain A and for all (x',y') and which is positively homogeneous in (x',y') for each (x,y) in A with $F(x,y,0,0)=0$. Let C: $x=x(s)$, $y=y(s)$, $0\leqq s\leqq l$, be a rectifiable curve and let

$$\Phi(\delta)=F[x(\alpha),y(\alpha),x(\beta)-x(\alpha),y(\beta)-y(\alpha)],\quad \delta=(\alpha,\beta).$$

The author then proves that the Burkill integral of $\Phi(\delta)$, which is the same as the Weierstrass integral $W_C(F)$, exists and is finite. The author indicates a simplification of his previous proofs of various lower semicontinuity theorems in the calculus of variations. *C. B. Morrey, Jr.*

Morse, Anthony P. and Randolph, John F. Gillespie measure. Duke Math. J. 6, 408–419 (1940). [MF 2324]

D. C. Gillespie suggested the following definition of linear measure for a plane set of points: Let A be the set, $U_1, U_2, \cdots$ a sequence of open convex regions with greatest diameter $d_i<\rho$ whose union contains A. Let c_i be the length of the simple closed curve bounding U_i, and $G_\rho(A)$ the greatest lower bound of $\sum c_i/2$. As ρ decreases $G_\rho(A)$ does not decrease. The limit (finite or infinite) as $\rho\to 0$ of $G_\rho(A)$ is the Gillespie outer measure $G^*(A)$ of the set A. The only difference between this definition and that of Carathéodory outer measure $L^*(A)$ is that in the latter d_i replaces $c_i/2$. However, the authors show that the properties of the two are strikingly different. $G^*(A)\leqq\pi L^*(A)/2$, and a set exists for which the equality holds. If $m^*(A^x)$, $m^*(A^y)$ are the outer Lebesgue measures of the projections of A on the x- and y-axes, then

$$G^*(A)\geqq\{[m^*(A^x)]^2+[m^*(A^y)]^2\}^{\frac{1}{2}}.$$

This relation does not hold for $L^*(A)$. A set is exhibited for which $m^*(A^x)=m^*(A^y)=L^*(A)=1$, $G^*(A)=2^{\frac{1}{2}}$. The set A is measurable if, for every set W, $G^*(W)=G^*(AW)+G^*(W-AW)$. If A is a rectifiable curve, then $G(A)$, the measure of A, is the length of A. The definition is extended to two dimensional sets in three dimensional space by replacing U_i by open convex three dimensional domains U_i, and c_i by the surface area of the boundary of U_i. If A is a measurable set in the xy-plane, and B the set: $(x,y)\subset A$, $0\leqq z\leqq h$, then the two dimensional measure of B is $hG(A)$. The corresponding relation has not yet been established for Carathéodory measure. *R. L. Jeffery.*

Carathéodory, C. Über die Differentiation von Massfunktionen. Math. Z. 46, 181–189 (1940). [MF 2394]

A method is sketched for defining abstractly an analogue of the derivative for the Massfunktionen introduced by the author in his abstract theory of integration. This makes possible the proof of a theorem on the recovery of an integrand by differentiation almost everywhere, analogous to Lebesgue's fundamental theorem. The full development with detailed proofs will appear in the second volume of the new edition of the author's "Reelle Funktionen" [the first part of which appeared in Leipzig, 1939].

P. Franklin (Cambridge, Mass.).

Cesari, Lamberto. Sul teorema di densità in senso forte. Ann. Scuola Norm. Super. Pisa (2) 8, 301–307 (1939). [MF 2043]

The paper contains proofs of the following known results:

(1) If E is a measurable plane set, then for almost every point (ξ,η) of E

$$\lim_{\delta(I)\to 0}\frac{\text{meas }(EI)}{\text{meas }I}=1,$$

where I is any interval with sides parallel to the axes and having (ξ,η) as center, and $\delta(I)$ denotes the diameter of I [Saks, Theory of the Integral, p. 129].

(2) If the function $f(x,y)$ is measurable, and the function $f(x,y)\log^+|f(x,y)|$ integrable over a rectangle R, then at almost every point (ξ,η) of R

$$\lim_{\delta(I)\to 0}\frac{1}{\text{meas }I}\iint_I f(x,y)dxdy=f(\xi,\eta),$$

where I and $\delta(I)$ have the same meaning as before [Jessen, Marcinkiewicz and Zygmund, Fund. Math. 25, 217–234 (1935)]. It is known that the latter result is false for functions f that are merely integrable over R [cf. the paper quoted last]. Here the author proves slightly more, namely:

(3) Let $\alpha(t)$ be any non-decreasing function, positive for $t>0$ and tending to 0 with t. There is then an integrable function $f(x,y)$ such that at every point (ξ,η) interior to R

$$\overline{\lim_{\delta(I)\to 0}}\frac{1}{\text{meas }I}\iint_I f(x,y)dxdy=+\infty,$$

even if the sides h,k of I satisfy the conditions $h\geqq k\alpha(k)$, $k\geqq h\alpha(h)$. *A. Zygmund* (South Hadley, Mass.).

Jeffery, R. L. Copeland's definition of a Stieltjes integral. Bull. Amer. Math. Soc. 46, 512–519 (1940). [MF 2420]

The Copeland definition of Stieltjes integral [Bull. Amer. Math. Soc. 43, 581–588 (1937)] is altered as follows: f being a monotone function with $f(\alpha+0)=0$, $f(\beta-0)=1$. Consider the double array x_{mn}, $m=1,2,\cdots,n$, on $\alpha<x<\beta$, where x_{kn} is the greatest lower bound of the numbers x for which $f(x-0)\leqq k/n\leqq f(x+0)$. If $g(x)$ is on $\alpha<x<\beta$ and $G_n=\sum_{m=1}^{n}g(\xi_{mn})/n$, where $x_{m-1,n}\leqq\xi_{mn}\leqq x_{mn}$, with suitable modifications for $m=1$ and n, then $CS\int_{\alpha<x<\beta}gdf$ exists if $\lim_n G_n$ exists independent of the choice of ξ_{mn}. In Copeland's definition $\xi_{mn}=x_{mn}$, the sequence x_{mn} is rearranged as a simple sequence $x_{11}, x_{12}, x_{22}, x_{13}, x_{23}, x_{33}, \cdots$ and the average of the values of $g(x)$ is taken to any place in this simple sequence. Jeffery's definition is related to that of Izumi [see the preceding review]. Essentially it amounts to subdividing the value space of $f(x)$ into a sequence of equal subdivisions. For $f(x)=x$ this reduces to the Riemann definition limited to the sequence of subdivisions with subintervals of equal length. It is proved that the Stieltjes integral so defined exists if and only if the ordinary Riemann Stieltjes interval of $\int gdf_c$ exists, where f_c is the continuous part of f, and that it is equal to $\int gdf_c+\sum_x g(x)[f(x+0)-f(x-0)]$. Also that $F(x)=\int^x gdf$ has a derivative dF/df equal to g excepting at a set of f-measure zero.

T. H. Hildebrandt (Ann Arbor, Mich.).

Foradori, Ernst. Zur Theorie des Carathéodoryschen Integrales. Deutsche Math. 4, 578–582 (1939). [MF 526]

The author shows that Teiltheorie includes Carathéodory's Somen [S.-B. Bayer. Akad. Wiss. 1938, 175–183] by proving that if K is an arbitrary collection of objects it satisfies Carathéodory's first two axioms if and only if (α) there exists in K a reflexive and transitive relationship $\leqq$. Moreover K will form a collection of Somen if and only if it

satisfies (α) and three additional axioms. In view of Carathéodory's paper this provides a somewhat simpler algebraic basis for the Lebesgue integral. *B. J. Pettis.*

Aronszajn, N. Quelques recherches sur l'intégrale de Weierstrass. II. Intégrale de Hellinger. Revue Sci. (Rev. Rose Illus.) **78**, 165–167 (1940). [MF 2198]

The author gives some additions to his previous theory [Revue Sci. (Rev. Rose Illus.) **77**, 490–493 (1939); these Rev. **1**, 111] of a generalized integral of functions of one real variable, and shows how this theory yields a generalization of the Hellinger integral. *L. M. Graves.*

Aronszajn, N. Quelques recherches sur l'intégrale de Weierstrass. III. Longueur généralisée. Revue Sci. (Rev. Rose Illus.) **78**, 233–239 (1940). [MF 2452]

[Cf. the preceding review.] The author gives various sufficient conditions for the existence of his generalized integral. One case is that when the integral is the Weierstrass generalization of a calculus of variations integral in Euclidean space. There follow extensions of this idea to normed vector spaces and to more general abstract spaces with a distance. Various ways are given for defining angles in abstract spaces. There is stated a generalization to abstract spaces of the theorem that a rectifiable curve has a tangent almost everywhere. Proofs are largely omitted. Some indications are given of relations with previous work of Menger, Pauc and Clarkson. *L. M. Graves.*

Tolstoff, G. La méthode de M. Perron pour l'intégrale de M. Denjoy. C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 470–472 (1939). [MF 2076]

The author gives a definition of a Perron integral which is equivalent to the general Denjoy integral, the proof of this equivalence to appear elsewhere. A function $\Phi(x)$ is called D-derivable on an interval (a, b) if (a, b) is the sum of a sequence of perfect sets $P_1, P_2, \cdots$ such that on each set Φ is derivable with respect to the set. It is shown that, if $i \neq j$, $\Phi'_{P_i} \neq \Phi'_{P_j}$ on at most a denumerable set of values of x. Except on this denumerable set, let $D\Phi(x) = \Phi'_{P_n}(x)$, P_n any one of the sets $P_1, P_2, \cdots$. A function $\varphi(x)$ is major to a function $f(x)$ if φ is continuous and D-derivable on (a, b), $\varphi(a) = 0$, $D\varphi(x) \geqq f(x)$, $D\varphi > -\infty$. The corresponding minor function is defined, and the definition of integrability follows from these in the usual way. The author compares this definition with those of Ridder [Fund. Math. **21**, 1–10, 11–19 (1933); **22**, 136–162, 163–179 (1934)]. A point of difference is that the derived numbers used by Ridder which correspond to the author's derivatives Φ'_{P_i}, Φ'_{P_j} may differ on a more than denumerable set, which is, nevertheless, of zero measure. *R. L. Jeffery* (Wolfville, N. S.).

Izumi, Shin-ichi. An abstract integral. II. Proc. Imp. Acad. Tokyo **16**, 87–89 (1940). [MF 2220]

Using ideas suggested by Schoenberg [Math. Z. **28**, 171–199 (1928)] in connection with the asymptotic distribution of real numbers, the Stieltjes integral $\int f(z)dz(x)$, with $z(x)$ monotone on $a \leqq x \leqq b$ and $z(b) - z(a) = 1$, $f(x)$ real valued, is defined as follows: The sequence (x_{nm}), $m = 1, \cdots, n$, will be called a sequence $(C, 1)$ if (a) $a \leqq x_{nm} \leqq b$, (b) for any subinterval (α, β) of (a, b) with $z(x)$ continuous at α and β we have $\lim_n M_n/n = z(\beta) - z(\alpha)$, where M_n is the number of x_{nm} in (α, β) for fixed n. Then $\int f dz$ exists if $\lim_n (1/n) \sum_{m=1}^{n} f(x_{nm})$ exists for all sequences $(C, 1)$. This definition is closely related to that of Copeland [Bull. Amer. Math. Soc. **43**, 581–588 (1937)], and it is asserted that it is equivalent to a definition of an extended Riemann Stieltjes integral due to Ridder [Prace Mat.-Fys. **41**, 65–95 (1934)]. Extension to the case when $f(x)$ is measurable on $a \leqq x \leqq b$ and $z(e)$ is a completely additive, positive set function on measurable sets with $z(E) = z(a, b) = 1$ is made by assuming that the (x_{nm}) satisfy for any set $e = E_x[\alpha \leqq f(x) < \beta]$ the condition $\lim_n N_n/n = z(e)$, where N_n is the number of points of x_{nm} in e, n fixed. Other extensions: (1) f on an abstract space E to a linear normed vector space E_1, where E has defined on it a Riemann field $\mathfrak{X}$ with a Jordan measure μ with $\mu(E) = 1$; (2) E_1 a linear metric space, E with a Lebesgue field $\overline{\mathfrak{X}}$, with $mE = 1$, f a $\overline{\mathfrak{X}}$ measurable function on E to E_1, that is, such that K a sphere in E_1 implies $E_x[f(x) \varepsilon K]$ belongs to $\overline{\mathfrak{X}}$. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Fréchet, Maurice. Sur une définition intrinsèque de l'aire d'une surface courbe comme limite d'aires polyèdrales inscrites. Ann. Scuola Norm. Super. Pisa (2) **8**, 285–300 (1939). [MF 2042]

The author first gives an "intrinsic" definition of an "inscribed polyhedron" in a surface S (in the sense of Fréchet) and of the "convergence" of a sequence of such polyhedra to S. These definitions are intrinsic in the sense that the parametric representations of S and the polyhedra are used only to determine the arrangement of the points on the surfaces. The definition of convergence is equivalent to Fréchet's previous definition which is now in general use.

The principal result of the paper is the following: Let S be a surface without multiple points which possesses at least one parametric representation $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ on a plane region Q bounded by a polygon, in which the functions x, y and z are of class C' with $EG - F^2 \neq 0$ on the closed region Q. Let $\{\Pi_n\}$ be any sequence of polyhedra tending to S in which there is a number γ, $0 < \gamma < \pi$, such that every angle of every face of each Π_n is less than γ. Then the areas of the Π_n tend to the Lebesgue area of S which is, in turn, given by the classical integral whenever S is represented parametrically as above. Various modifications of this result are also given. It is pointed out that this result does not necessarily hold for any more general surfaces S, as it does not even hold for a general simple polyhedron S. *C. B. Morrey, Jr.* (Berkeley, Calif.).

Theory of Functions of Complex Variables

Poli, l'Abbé L. Sinus du $n^{\text{ième}}$ ordre et calcul symbolique. Ann. Soc. Sci. Bruxelles. Sér. I. **60**, 15–30 (1940). [MF 2536]

Generalizing the ordinary sine and cosine, the author studies the integral functions $f_i(x)$ whose Laplace transforms are $p^i/(p^n+1)$, $i = 1, \cdots, n$. *W. Feller.*

Privaloff, I. I. On integrals of Cauchy type. Doklady Akad. Nauk SSSR (N.S.) **23**, 859–862 (1939). (Russian) [MF 1962]

Let L be a closed contour made up of a finite number of arcs, each with a continuously turning tangent. The exterior angle of L at the point x_s is denoted by $\pi\theta$, where $-1 < \theta < 1$ (spines, which correspond to $\theta = \pm 1$, are excluded). The author proves that, if $f(z)$ satisfies a Lipschitz condition of order α $(0 < \alpha \leqq 1)$ on L, then

$$\frac{1}{2\pi i}\int_L \frac{f(x)dx}{x-z} \to \frac{1}{2\pi i}\int_L \frac{f(x)dx}{x-x_s} \pm \frac{1 \mp \theta}{2} f(x_s),$$

as $z \to x_0$ on any non-tangential path; the upper or lower signs are taken according as $z \to x_0$ from inside or outside L; the integral on the right is a Cauchy principal value at x_0. The author proves further that the integral on the right also satisfies a Lipschitz condition, of order α if $0<\alpha<1$, and of order $1-\eta$ for any positive η if $\alpha=1$. The first result is deduced by conformal mapping from an older result of the author for the case where L has a tangent at x_0 (and $f(z)$ is less restricted) [Privaloff, The Cauchy Integral, 1918, pp. 63–64 (Saratoff thesis, in Russian)]. The second result is proved directly [cf. the author's theorem on conjugate functions: Zygmund, Trigonometrical Series, 1935, p. 156]. The corresponding results for the potential of a double layer on L are given. *R. P. Boas, Jr.* (Durham, N. C.).

Fédoroff, V. S. Sur les suites des intégrales curvilignes. Rec. Math. [Mat. Sbornik] N.S. **6** (**48**), 53–65 (1939). (Russian. French summary) [MF 1434]

Let $f(z)$ be a single-valued and continuous function of the complex variable z in a simply connected region Δ and set

$$I(L)=\int_L f(z)dz,$$

the integration being performed in the positive sense along L, a simple, closed, rectifiable curve lying in Δ. The author studies the behavior of sequences of line integrals of the above type. Denote by $\delta(L)$ the region bounded by L, by (ζ, α) the circle $|z-\zeta|<\alpha$, and by $\delta(L, \alpha)$ the region formed as a sum of all circles (ζ, α) as ζ describes the curve L. The author writes "$L_n \to L$ from inside" if the sequence of curves L_n possesses the properties: (1) all L_n are interior to $\delta(L)$ and each L_n is interior to $\delta(L_{n+1})$; (2) for every $\epsilon>0$ there exists an integer $N(\epsilon)>0$ such that for $n>N(\epsilon)$ all curves L_n belong to $\delta(L, \epsilon)$.

A functional $I(L)$ is called bounded in some region if for all curves L in this region $|I(L)|<c$, where c is a constant independent of L. Finally, the author writes "$L_n \to \zeta$," where ζ is some point, if for every $\epsilon>0$ there exists an integer $N(\epsilon)>0$ so that for $n>N(\epsilon)$ all curves L_n belong to (ζ, ϵ). Two theorems are proved: (I) Let $L_n \to L_0$ from inside. A necessary and sufficient condition that $I(L_n) \to I(L_0)$ for any such sequence L_n is the boundedness of $I(L)$ for all L in some arbitrarily small neighborhood of every point of L_0 and lying in $\delta(L_0)$. (The boundedness is not necessarily uniform for all points of L_0.) (II) Generalization of Morera's Theorem: let E be a perfect, everywhere discontinuous set of points in Δ. Let $f(z)$ possess the properties: (1) $I(L_n)$ cannot become infinite as $n \to \infty$ for any sequence of curves L_n which contain in their interior any point of the set E, if $L_n \to$ this point and L_{n+1} lies in $\delta(L_n)$ for $n=1, 2, \cdots$; (2) $I(L)=0$ if L lies in Δ and does not pass through any points of the set E (although it may contain such points in its interior). Then, $f(z)$ is analytic in Δ. The proofs are largely based on considerations of continuity.

W. Seidel (Rochester, N. Y.).

Lindemann, F. Zur Theorie der konformen Abbildung. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1939**, 27–67 (1939). [MF 1564]

The author derives formulas for the conformal mapping of a simply connected domain on a half-plane containing only quadratures. His conclusions and formulas are incorrect. *C. Loewner* (Hanover, N. H.).

Golusin, G. M. Iterationsprozesse für konforme Abbildungen mehrfachzusammenhängender Bereiche. Rec. Math. [Mat. Sbornik] N.S. **6** (**48**), 377–382 (1939). (Russian. German summary) [MF 1945]

A new method of iteration is introduced for the conformal mapping of any finitely connected region on a plane with finite, rectilinear and parallel slits forming an angle α with the positive axis of reals. This method of iteration is based on the following idea. Being given a closed, bounded set of points, the author defines its width in the direction α as the width of the narrowest strip containing this set and bounded by straight lines forming the angle α with the axis of reals. By making use of Bieberbach's theorem of areas it is then shown that, if B, a simply connected region in the z-plane whose boundary lies in the circle $|z|<R$ and has a width in the direction α not less than h_0, where $h_0>0$, is mapped conformally on the ζ-plane with a rectilinear slit making an angle α with the real axis by means of the function

$$\zeta=z+\frac{\alpha_1}{z}+\cdots,$$

then $\Re(e^{-2\alpha i}\alpha_1) \geqq \delta$, where $\delta=\delta(R, h_0)$ is a positive constant, independent of B. Now, let B be an n-tuply connected region in the z-plane containing the point $z=\infty$ in its interior and bounded by the continua $c_1, c_2, \cdots, c_n$ of which c_k has the greatest width h in the direction α. Map the simply connected region containing $z=\infty$ in its interior and bounded only by the continuum c_k on the z_1-plane cut along a slit making an angle α with the axis of reals. Thereby $c_1, c_2, \cdots, c_{k-1}, c_{k+1}, \cdots, c_n$ go over into the continua $c_1', c_2', \cdots, c_{k-1}', c_{k+1}', \cdots, c_n'$. Of these, let c_{k_1}' have the greatest width h_1 in the direction α. The above process is repeated now with the continuum c_{k_1}', and so forth. It is shown by making use of the above lemma and the general method of normal families that this process converges in the limit and gives the desired map of B on a slit region. Similar processes are indicated to obtain maps on canonical regions formed by cutting the plane along logarithmic spirals.

W. Seidel (Rochester, N. Y.).

Cooke, Richard H. On Taylor series for which $\lim_{n\to\infty} a_{n+1}/a_n=1$. Tôhoku Math. J. **46**, 319–327 (1940). [MF 2445]

The author proves the following results: (I) Let

$$f(z)=\sum_{n=0}^{\infty} a_n z^n=\sum_{n=0}^{\infty} e_n\left(\frac{z}{1-z}\right)^n$$

be a uniform function having $z=1$ as an isolated essential singularity. Also let $z=1$ be the sole singularity of $f(z)$ in the entire plane, and let $e_n \geqq 0$ for all n. Then $\lim_{n\to\infty} a_{n+1}/a_n=1$. (II) If $R_n(z)$ is the remainder after the term $a_n z^n$ in $\sum_{n=0}^{\infty} a_n z^n$ and if $\lim_{n\to\infty} a_{n+1}/a_n=1$, then $R_n(z) \sim a_n z^{n+1}/(1-z)$ uniformly for $|z| \leqq r<1$. An extension of this latter result is also given. *N. Levinson* (Cambridge, Mass.).

Denjoy, Arnaud. Sur les séries de Taylor admettant leur cercle de convergence comme coupure. J. Math. Pures Appl. **19**, 45–49 (1940). [MF 1990]

The author gives examples of Taylor's series $\sum a_n z^n$, $a_n \geqq 0$, having the circle of convergence as a natural boundary, and where the a_n depend on an infinite number of parameters. Moreover there exist Taylor series $\sum c_n z^n$ such that $a_n=c_n \ell_n$ but such that $\sum c_n z^n$ is analytically continuable over the whole plane and has as its only singular points a perfect set of points of measure zero. *N. Levinson.*

Gelfond, A. On the Taylor series associated with an integral function. C. R. (Doklady) Acad. Sci. URSS (N.S.) 23, 756–758 (1939). [MF 1957]

The author generalizes the well-known relationship between

$$F(z)=\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n \quad \text{and} \quad f(z)=\sum_{n=0}^{\infty}\frac{a_n}{z^{n+1}}$$

by obtaining a relationship between $F(z)$ and $f_\alpha(z)$, where

$$f_\alpha(z)=\sum_{n=0}^{\infty}\frac{e^{\frac{1}{2}n(n-1)\alpha i}a_n}{z^{n+1}}.$$

By using $f_\alpha(z)$ he is enabled to prove two theorems. One of these concerns the structure of an entire function of exponential type $F(z)$ which satisfies $F^{(n)}(e^{in\alpha})=0$, $n=1, 2, \cdots$. The other concerns the class of solutions of a certain functional equation. *N. Levinson* (Cambridge, Mass.).

Shah, S. M. A theorem on integral functions of integral order. J. London Math. Soc. **15**, 23–31 (1940). [MF 2580]

The author shows that if $f(z)$ is a canonical product of order and genus p ($p=0, 1, 2, \cdots$), and $M(r)$ and $n(r)$ denote respectively the maximum of $|f(z)|$ and the number of zeros of $f(z)$ in $|z|\leqq r$, then

$$\liminf_{r\to\infty}\frac{\log M(r)}{n(r)\varphi(r)}=0 \tag{1}$$

for every positive continuous non-decreasing $\varphi(x)$ such that

$$\int^{\infty}\frac{dx}{x\varphi(x)} \tag{2}$$

converges. This, the author states, was suggested by the known theorem that if $f(z)$ is an entire function of non-integral order,

$$\liminf_{r\to\infty}\frac{\log M(r)}{n(r)}<\infty;$$

this result fails for functions of integral order. The convergence of (2) is not a necessary condition for the theorem; the author constructs for each positive integer p a positive continuous non-decreasing $\varphi(x)$ such that (2) diverges, while (1) is satisfied for every entire function of order p. [The function $\varphi(x)=\log x \log\log x$ $(x>e)$ has the same properties with respect to functions of zero order.]

R. P. Boas, Jr. (Durham, N. C.).

Helmer, Olaf. Divisibility properties of integral functions. Duke Math. J. 6, 345–356 (1940). [MF 2319]

The author denotes by K a subfield of the field of complex numbers, by $K[z]$ the domain of polynomials in z with coefficients in K, by $K\langle z\rangle$ the domain of integral functions whose Maclaurin series have coefficients in K, and by $K^*\langle z\rangle$ the subset of $K\langle z\rangle$ consisting of all its elements which are functions of finite order. He determines the units and irreducible elements of $K\langle z\rangle$; establishes the unique decomposition into irreducible factors of elements of $K\langle z\rangle$; and shows that, while $K\langle z\rangle$ contains ideals which are not principal, every ideal in $K\langle z\rangle$ with a finite basis is principal. The divisibility theory of $K\langle z\rangle$ has the unsatisfactory feature that an irreducible element of $K[z]$ may be reducible in $K\langle z\rangle$. This situation cannot arise in $K^*\langle z\rangle$, as follows from the following theorem which the author proves: If K_2 is a field over K_1, $F(z)$ is in $K_2[z]$ but not in $K_1[z]$, and $F(0)=1$, then no element of $K_1^*\langle z\rangle$ has precisely the same zeros as $F(z)$. The author promises a future discussion of the divisibility theory of $K^*\langle z\rangle$, in which the results are less simple and consequently of greater interest than in $K\langle z\rangle$.

R. P. Boas, Jr. (Durham, N. C.).

Helmer, Olaf. Theorems of the Picard type. Duke Math. J. 6, 38–47 (1940). [MF 1541]

L'auteur se propose d'apporter quelques compléments aux théorèmes classiques de Picard et de Borel, en se bornant au cas des fonctions entières d'ordre fini. La méthode utilisée est celle de Borel [Leçons sur les fonctions entières, 2e édition]. Les résultats sont du type suivant: Soient $f(z)$ et $g(z)$ deux fonctions entières du même ordre fini ρ, et $A(z)$ et $B(z)$ deux fonctions entières arbitraires d'ordre inférieur à ρ et définies à un facteur (fonction entière) près. L'exposant de convergence des zéros de $Af+Bg$ est toujours moindre que ρ si f et g sont les produits de fonctions d'ordre inférieur à ρ par une même exponentielle $e^{H(z)}$ d'ordre ρ; dans les autres cas, cet exposant de convergence est égal à ρ sauf au plus pour deux couples de fonctions A, B. Cette proposition, ainsi que d'autres qui ne se ramènent pas au théorème de Borel sur les fonctions entières, rentre dans le cas du théorème de Borel sur les fonctions méromorphes. (On applique le théorème de Borel [Leçons sur les fonctions méromorphes, chap. 3] au quotient $f:g$.)

G. Valiron (Paris).

Anastassiadis, Jean. Sur les valeurs exceptionnelles des fonctions entières et méromorphes d'ordre fini. C. R. Acad. Sci. Paris **210**, 204–206 (1940). [MF 1634]

A statement of results. If $\sum a_n z^n=a+\exp(\alpha_0+\alpha_1 z+\cdots+\alpha_p z^p)$, the exceptional value a can be expressed as a rational function of $a_0, a_1, \cdots, a_{2p}$. The two possible exceptional values of the meromorphic function $\sum a_n z^n/\sum b_n z^n$ of order not greater than p can be exhibited as the roots of a quadratic equation rational in $a_0, a_1, \cdots, a_{2p-1}, b_0, b_1, \cdots, b_{2p-1}$.

A. J. Macintyre (Aberdeen).

Laurent-Schwartz, Mme. Sur une propriété de la fonction $m(r, A)$ de M. Nevanlinna dans les fonctions méromorphes. C. R. Acad. Sci. Paris **210**, 525–526 (1940). [MF 2261]

Consider a meromorphic function. If A is a regular value of the inverse function, or more generally if the number of branch values of the function which have branch points in an arbitrarily small but fixed region about A is finite, then $m(r, A)$ is bounded; in particular A is not a deficient value of the original function. *N. Levinson.*

Hibbert, Lucien. La notion de cassure au point singulier essentiel des courbes d'égal module et d'égal argument. C. R. Acad. Sci. Paris **210**, 499–501 (1940). [MF 2014]

This paper is another in a series of articles by the author in which he studies the topological properties of the net of curves $|f|=$constant, $\arg f=$constant, where f is an analytic function in the entire plane except at a point A, where it has an essential singularity. [C. R. Acad. Sci. Paris **209**, 783–786 (1939); **210**, 35–37 (1940); these Rev. **1**, 211]. By defining a method of selecting a continuation of a given branch at a singular point of the net, he is able to consider regions bounded by curves of the net, and he studies these regions in detail, especially in the neighborhood of A, determining several types of behavior and producing examples. He also points out how the asymptotic behavior of F at A is related to the topological behavior of the net at A.

J. W. Green (Rochester, N. Y.).

Basilevitch, I. Sur un théorème de Littlewood et Paley. Rec. Math. [Mat. Sbornik] N.S. 6 (48), 337–344 (1939). (Russian. French summary) [MF 1943]

Littlewood and Paley proved [J. London Math. Soc. 7, 167–169 (1932)] that, if the function

$$f(z)=z+c_2z^2+\cdots+c_{2n-1}z^{2n-1}+c_{2n}z^{2n}+\cdots$$

is regular and univalent for $|z|<1$, and if $c_2=c_4=c_6=\cdots=0$, then there is an absolute constant P such that

$$|c_n|\leqq P, \qquad \text{for } n=1, 2, \cdots. \tag{1}$$

In the present paper the author considers the decomposition $f(z)=f_1(z)+f_2(z)$, where $f_1(z)=z+c_3z^3+\cdots$ is the odd, and $f_2(z)=c_2z^2+c_4z^4+\cdots$ the even part of $f(z)$, and asks for conditions concerning $f_2(z)$ which would lead to the inequalities (1). The following results are obtained: (i) If $f(z)$ is regular and univalent for $|z|<1$, and if

$$|f_2'(z)|\leqq A(1-|z|)^{-1+\epsilon}, \qquad \epsilon>0;\ |z|<1, \tag{2}$$

then

$$|f(z)|\leqq K(1-|z|)^{-1}, \qquad |z|<1, \tag{3}$$

where K depends only on A and ϵ. (ii) If $f(z)$ is regular and univalent for $|z|<1$ and satisfies the condition (3), then we have (1), with P depending on K only. From (i) and (ii) follows at once that (iii) if $f(z)$ is regular and univalent for $|z|<1$, and if the even component $f_2(z)$ of $f(z)$ satisfies (2), then the coefficients of $f(z)$ satisfy (1) with P depending on A and ϵ only. (iv) If $f(z)$ is regular and univalent for $|z|<1$, and if $|c_{2n}|\leqq An^{-1+\epsilon}$ $(\epsilon>0)$, then the coefficients c_n satisfy (1) with P depending on A and ϵ only. *A. Zygmund.*

Golusin, G. M. Zur Theorie der schlichten Funktionen. Rec. Math. [Mat. Sbornik] N.S. 6 (48), 383–388 (1939). (Russian. German summary) [MF 1946]

By giving a generalization of Löwner's well-known parametric representation of univalent functions, the author proves the following results: Let the functions $f_1(z)$ and $f_2(z)$, $f_1(0)=f_2(0)=0$, $f_1'(0)>0$, $f_2'(0)>0$ be regular and univalent in $|z|<1$ and let $\zeta=f_1(z)$ map $|z|<1$ on a region B_1 and $\zeta=f_2(z)$ map $|z|<1$ on a region B_2 which is contained in B_1, but not identical with B_1. Then, $|f_2(z)|<|f_1(z)|$ in the circle $0\leqq|z|<r_0$, where r_0 is the root of the equation

$$\log\frac{1+r}{1-r}+2\operatorname{arctg} r=\frac{\pi}{2}.$$

The number r_0 can not be replaced by a larger number. Furthermore, $|f_2'(z)|<|f_1'(z)|$ in the circle $|z|<1/10$. The largest circle in which the latter inequality holds is not known. *W. Seidel* (Rochester, N. Y.).

Joh, Kenzo. Theorems on "schlicht" functions. IV. Proc. Phys.-Math. Soc. Japan (3) 22, 329–343 (1940). [MF 2371]

Employing the principle of harmonic measure due to R. Nevanlinna, the author gives new proofs of two recent theorems by H. Unkelbach [Jber. Deutsch. Math. Verein. 49, 38–49 (1939)]. The first of these theorems is to the effect that if $W=F(z)$ be regular for $|z|<R$, $F(0)=0$, $F(z)\neq1$ for $|z|<R$, $F(z)$ univalent and convex for $|z|<R$, then the Lebesgue measure α of the set of points on $|z|=R$ for which $|F(z)|\geqq R$ satisfies the inequality $\alpha<8$. And for any arbitrary given small positive number ϵ, there exists a value of R and a function such that $\alpha>8-\epsilon$. If instead of being convex $F(z)$ is univalently star-like for $|z|<R$ in addition to the normalization mentioned above, the second theorem establishes the fact that the Lebesgue measure β of the set of points on $|z|=R$ for which the radial limiting value $\lim_{|z|\to R}|F(z)|\geqq R^2$ is such that $\beta<8$ and again the number 8 cannot be replaced by a smaller one. Unkelbach gave a first proof of these theorems using as a basis Löwner's lemma which is that, if $W=\phi(z)$ be a function regular for $|z|<R$, $\phi(0)=0$, $|\phi(z)|\leqq R$ and $|\phi(z)|=R$ continuously on an arc A on $|z|=R$ of length $\alpha_{(z)}$, and if $\alpha_{(W)}$ be the length of the arc corresponding to the arc A, then $\alpha_{(W)}\geqq\alpha_{(z)}$. Unkelbach conjectured that the second theorem is correct if the normalized $F(z)$ is merely univalent (instead of univalently star-like). The difficulties of this conjecture were overcome by the author through the powerful principle of harmonic measure which appears to be natural and essential to the proof. A further extension is obtained for multivalent functions $F(z)$. *M. S. Robertson.*

Loomis, Lynn H. The radius and modulus of n-valence for analytic functions whose first $n-1$ derivatives vanish at a point. Bull. Amer. Math. Soc. 46, 496–501 (1940). [MF 2418]

In this paper the following class of functions is considered: for $|z|\leqq1$, $f(z)=z^n+a_{n+1}z^{n+1}+\cdots$ is analytic and satisfies $|f(z)|\leqq M$ $(M>1)$. The radius of n-valence $\rho=\rho(n, M)$ is the radius of the largest circle with center at $z=0$ within which no member of the class can assume any value more than n times. The modulus of n-valence is the radius of the largest circle with center at $w=0$ whose interior is covered n times by the map of $|z|<\rho$ under $w=f(z)$ for every function belonging to the given class. The author finds the exact value of the radius of n-valence and the modulus of n-valence as functions of n and M. The proof makes use of a theorem of Ozaki and various applications of the Schwarz lemma. The case $n=1$ has been treated by Landau and Dieudonné. *A. C. Schaeffer* (Stanford University, Calif.).

Macintyre, A. On a theorem concerning functions regular in an annulus. Rec. Math. [Mat. Sbornik] N.S. 5 (47), 307–308 (1939). (Russian. English summary) [MF 2301]

If $f(z)$ is regular for $1<|z|<R$, uniform and never zero, and $\ln f(z)$ increases by $2\pi i$ as z moves round this annulus, then

$$\max_{0\leqq\theta\leqq2\pi}\left|\frac{f(Re^{i\theta})}{f(e^{i\theta})}\right|\geqq R.$$

This result is a slight modification of one due to Pólya. *Author's summary.*

Kryloff, W. Über Funktionen, die in der Halbebene regulär sind. Rec. Math. [Mat. Sbornik] N.S. 6 (48), 95–138 (1939). (Russian. German summary) [MF 1437]

Let $F(w)$ be analytic in the unit circle $|w|<1$ and such that either

$$\int_{-\pi}^{\pi}|F(\rho e^{i\theta})|^p d\theta\leqq M^p<\infty, \qquad \rho<1,\ p>0,$$

or

$$\int_{-\pi}^{\pi}\log^+|F(\rho e^{i\theta})|\,d\theta\leqq M<\infty, \qquad \rho\leqq1,$$

where M does not depend on ρ. The corresponding classes of functions are denoted respectively by H_p and N. On the other hand the classes $\mathfrak{H}_p$ and $\mathfrak{N}$ consist of functions $f(z)$, analytic on the half plane $y=\Im z>0$ and such that, re-

spectively,

$$\int_{-\infty}^{\infty} |f(x+iy)|^p dx \leqq M^p < \infty, \qquad y>0,$$

or

$$\int_{-\infty}^{\infty} \log^+ |f(x+iy)|\, dx \leqq M < \infty, \qquad y>0,$$

where M does not depend on y. In the case $p \geqq 1$ various properties of functions of class $\mathfrak{H}_p$, analogous to those of H_p, were established by Hille and Tamarkin [Ann. of Math. (2) 34, 606–614 (1933); Fund. Math. 25, 329–352 (1935)]. Their results, by an entirely different method, were extended to the case $p>0$ by Kawata [Jap. J. Math. 13, 483–491 (1937)]. In the present paper the author gives a unified and considerably simplified treatment of these properties in the case of any $p>0$, and, in addition, establishes a formula of parametric representation of functions of class $\mathfrak{H}_p$, analogous to the well-known formula for functions of the class H_p. The simplification became possible by using the important fact that the class H_p admits of another definition which is invariant under all conformal transformations, namely as the class of functions analytic in the unit circle and for which $|F(w)|^p$ admits of a harmonic majorant. In the same way N is the class of functions analytic in the unit circle and such that $\log^+ |F(\omega)|$ admits of a harmonic majorant. Using this fact the author is able to carry through an analogous discussion for functions of class $\mathfrak{N}$ and to give their parametric representation. Finally he discusses functions which can be represented by absolutely convergent Cauchy integrals, and obtains, among others, the following theorem: In order that $F(z)$, analytic in the half-plane $y>0$, would have limiting values $F(t)$ almost everywhere on the axis of reals and would be represented in the half-plane $y>0$ by the absolutely convergent Cauchy integral

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(t)}{t-z}\, dt,$$

it is necessary and sufficient that (i) $F(z) \in H$, and (ii) $F(t)(t+i)^{-1} \in L_1(-\infty, \infty)$. *J. D. Tamarkin.*

Strodt, Walter. On zeros of irregular Taylor's series, and an approximation problem. Ann. of Math. (2) 41, 350–355 (1940). [MF 1817]

If $f(z)=\sum_0^\infty b_n z^{\lambda_n}+h(z)$ is an integral function of order ρ with $\lambda_{n+1}>\lambda_n^\sigma$ $(\sigma>1)$ and $h(z)$ of order $\rho'<\rho$, then, writing $n(r)$ for the number of zeros of $f(z)$ in $|z|<r$, it is known that

$$(*) \qquad \overline{\lim_{r\to\infty}} \frac{\log n(r)}{\log r} = \rho, \quad \underline{\lim_{r\to\infty}} \frac{\log n(r)}{\log r} \leqq \max(\rho', \rho/\sigma)$$

[J. M. Whittaker, J. London Math. Soc. 8, 20–27 (1933)]. The second result is applied, along with the general theory of completeness and closure $L_2(0<\theta<2\pi)$ on a circle $z=Re^{i\theta}$, to establish an approximation theorem supplementing work of Gelfond [Rec. Math. [Mat. Sbornik] N.S. 4 (46), 149–156 (1938)]. If $f(z)$ is of the above type, $g(z)$ an integral function satisfying $g^{(n)}(0)=0$ if $f^{(n)}(0)=0$, then $g(z)$ can be approximated uniformly in any bounded domain by sums $\sum_0^N c_n f(a_n z)$ if a_n is a given sequence, which, arranged in order of $|a_n|$, satisfies

$$\overline{\lim_{n\to\infty}} \frac{\log n}{\log |a_n|} > \max(\rho', \rho/\sigma).$$

A proof of the inequality $(*)$ is given by using a special and direct theory of such an approximation.

A. J. Macintyre (Aberdeen).

Walsh, J. L. and Sewell, W. E. Note on degree of trigonometric and polynomial approximation to an analytic function, in the sense of least pth powers. Bull. Amer. Math. Soc. 46, 312–319 (1940). [MF 1839]

Let $w(\theta), f(\theta)$ be periodic with period 2π, $w(\theta)>0$ and continuous, and let

$$s_n(\theta) = \tfrac{1}{2}a_{n0} + \sum_{\nu=1}^{n} (a_{n\nu}\cos\nu\theta + b_{n\nu}\sin\nu\theta),$$

$$\int_{-\pi}^{+\pi} w(\theta)\,|f(\theta)-s_n(\theta)|^p d\theta < M\rho^{-np}n^{-(k+\alpha+1)p},$$

$0<\alpha\leqq 1$, $p>0$, $\rho>1$, $k\geqq 0$ integer, M independent of n. Then

$$\lim_{n\to\infty} \{\tfrac{1}{2}a_{n0} + \tfrac{1}{2}\sum_{\nu=1}^{n} [(a_{n\nu}+ib_{n\nu})z^{-\nu} + (a_{n\nu}-ib_{n\nu})z^{\nu}]\} = F(z)$$

exists and $F(e^{i\theta})=f(\theta)$ almost everywhere (everywhere if $f(\theta)$ is continuous). The function $F(z)$ is analytic in $\rho^{-1}<|z|<\rho$ and the kth derivative satisfies a condition of Lipschitz character. This result, dealing with approximation in the sense of least pth powers, is the analogue of a former result of the authors dealing with Tchebycheff approximation [Bull. Amer. Math. Soc. 44, 865–873 (1938)]. Similar approximations on the segment $(-1, +1)$ and in domains bounded by confocal ellipses are also considered. Finally a converse of the theorem above is indicated. The condition regarding the weight function can be generalized.

G. Szegö (Stanford University, Calif.).

Sewell, W. E. Continuity and degree of approximation by rational functions. Revista Ci., Lima 41, 435–451 (1939). [MF 1641]

The problem is that of the relation between degree of approximation to a given function by rational functions on a closed region or point set in the complex plane, and properties of continuity and differentiability on the boundary, the given function being analytic, by hypothesis or by demonstration, in the interior of the region. Specified degrees of approximation by rational functions imply properties of continuity on the boundary of the region to which the hypotheses relate or on the boundary of a larger region; and boundary properties imply specified degrees of approximation by rational functions having poles at preassigned points. Known theorems on polynomial approximation are brought to bear on the new problem through the medium of approximation to rational functions by means of polynomials, or approximation to polynomials by means of rational functions with prescribed poles. *D. Jackson.*

Walsh, J. L. On the degree of convergence of sequences of rational functions. Trans. Amer. Math. Soc. 47, 254–262 (1940). [MF 1585]

Let C_1 and C_2 be Jordan curves, C_2 in the interior of C_1; we denote by $\alpha_{n\nu}$, $1\leqq\nu\leqq n$, arbitrary points in the exterior of C_1, and by $\beta_{n\nu}$, $1\leqq\nu\leqq n+1$, arbitrary points in the interior of C_2. Let

$$\lim_{n\to\infty} \left| A_n \frac{(z-\beta_{n1})(z-\beta_{n2})\cdots(z-\beta_{n,n+1})}{(z-\alpha_{n1})(z-\alpha_{n2})\cdots(z-\alpha_{nn})} \right|^{1/n} = \phi(z),$$

with proper constants A_n, exist uniformly on every closed set between C_1 and C_2. Finally let $\phi(z)$ be constant on C_1 and C_2. We denote the curve $\phi(z)=\gamma$ by C_γ, $C_{\gamma_1}=C_1$, $C_{\gamma_2}=C_2$, $\gamma_2<\gamma_1$. In his Colloquium Publication [vol. 20, 1935, § 8.3] the author considers the sequence of rational

functions $r_n(z)$ of degree n associated with an analytic function $f(z)$ such that the poles of $r_n(z)$ are at $\alpha_{n\nu}$ and $r_n(\beta_{n\nu}) = f(\beta_{n\nu})$. Assuming that γ is the largest value such that $f(z)$ is analytic in the interior of C_γ, he proves that

$$\limsup_{n\to\infty} \{\max |f(z)-r_n(z)|, z\varepsilon C_\lambda\}^{1/n} \leqq \lambda/\gamma, \quad \gamma_2 < \lambda < \gamma.$$

The main purpose of the present paper is to show that in this inequality the sign = holds. This fact leads to the definition of the maximal convergence for sequences of rational functions, a concept previously introduced and discussed by the author for sequences of polynomials. The construction of proper sequences satisfying the above conditions, by considerations of the potential theory, is discussed. Finally the main results of the paper are generalized by an arbitrary conformal mapping, and the maximal convergence of the known special approximations is studied.

G. Szegö (Stanford University, Calif.).

Walsh, J. L. Note on the degree of convergence of sequences of analytic functions. Trans. Amer. Math. Soc. **47**, 293–304 (1940). [MF 1586]

By use of general principles on subharmonic functions, the author deals with the following theorem. Let R be a region whose boundary consists of a finite number of Jordan arcs J_k, and let $F_n(z)$ be analytic in R. Let $V(z)$ be harmonic and bounded in R and constant on each J_k. Then the supposition

$$\limsup_{n\to\infty} \{\text{l.u.b.}|F_n(z)|, z\varepsilon J_k\} \leqq \{e^{V(z)}, z\varepsilon J_k\}, \quad \text{for each } k,$$

implies the same inequality for an arbitrary closed domain Q (instead of J_k) in the interior of R. If the sign = holds for a particular Q, then it does for every Q. This is applied to the proof of a theorem on maximal convergence of sequences of polynomials. *G. Szegö.*

Lammel, Ernst. Über Approximation regulärer Funktionen eines komplexen Argumentes durch rationale Funktionen. Math. Z. **46**, 104–116 (1940). [MF 1484]

Let B be a domain bounded by a Jordan curve C, a in B and $\{b_n\}$ a sequence of points on C. If $f(z)$ is a given analytic function regular in B, the coefficients of the expansion

$$A_0 + \sum_{n=1}^{\infty} A_n \frac{(z-a)^n}{(z-b_1)(z-b_2)\cdots(z-b_n)}$$

can be determined such that the nth partial sum $s_n(z)$ satisfies the condition $s_n^{(\nu)}(a) = f^{(\nu)}(a)$, $0 \leqq \nu \leqq n$. The author obtains necessary and sufficient conditions for the uniform convergence of this expansion in every closed domain in B for every $f(z)$. One form of this condition is the following. Let $\{\beta_n\}$ be the sequence of the points on the unit circle $|w|=1$ in which $\{b_n\}$ is transformed by the mapping $w=\varphi(z, a)$ of B onto $|w|<1$ so that $z=a$ and $w=0$ correspond. Then $\{\beta_n\}$ must be uniformly distributed on $|w|=1$.

G. Szegö (Stanford University, Calif.).

Delange, Hubert. Sur la convergence des séries de polynomes de la forme $\sum a_n P_n(z)$ et sur certaines suites de polynomes. Ann. École Norm. **56**, 173–275 (1939). [MF 1581]

The present review indicates only the principal results of this remarkable paper. The general scope is to establish a relationship between the asymptotic distribution of the roots of $\{P_n(z) = z^n + \cdots\}$ and the asymptotic behavior of $n^{-1}\log|P_n(z)|$ for large n. Applications are given to series of the form $\Sigma = \sum a_n P_n(z)$. (A) The simple case is considered in which the zeros of $P_n(z)$ are in the interior of a given curve C. Let $\overline{\lim}\,|a_n|^{1/n} < (2r)^{-1}$, where $2r$ is the diameter of a circle containing C; then the series Σ is convergent in the interior of a certain curve Γ and divergent in the exterior of Γ. In order that all these curves Γ constitute a family of the type $F(x, y) = \text{const.}$, it is necessary and sufficient that $n^{-1}\log|P_n(z)|$ approach a limit $U(z)$ as $n\to\infty$ for all z in the exterior of C. A study of this (harmonic) function $U(z)$ follows. (B) In case C is the segment $(-1, +1)$, the last condition in A is equivalent to the convergence of the set-functions $\psi_n(e)\to\psi(e)$, where $\psi_n(e)$ designates the number of the zeros of $P_n(z)$ on the set e; furthermore,

$$U(z) = \int_{-1}^{+1} \log r \, d\psi(e),$$

where r is the distance of z from a variable point on $(-1, +1)$. This result is applied to the class of orthogonal polynomials defined by Szegö [Math. Ann. **82**, 188–212 (1921)]; the asymptotic behavior exterior to $(-1, +1)$ is known in this case. Also polynomials satisfying a recurrence formula of three members are considered in the light of a theorem of Poincaré. (C) The rest of the paper deals with the more general situation in which $P_n(z) = z^{p_n} + \cdots$, $p_n = O(n)$, with zeros whose set E is bounded and regularly distributed in the sense that $\lim \psi_n(e) = \psi(e)$ exists for each set e. Then for an arbitrary subsequence $\{n_\nu\}$

$$\overline{\lim}\, n_\nu^{-1} \log|P_{n_\nu}(z)| = \int_{E'} \log r \, d\psi(e),$$

with the possible exception of a subset of E' which can be covered by a set of circles, the sum of the αth powers of whose radii is less than ϵ (α, ϵ arbitrary). On this set the sign $<$ holds instead of the sign = of the last formula. If z is exterior to E', the full sequence $n^{-1}\log|P_n(z)|$ approaches the integral mentioned. (D) The assumption of the boundedness of E is waived. In this case the result of C has to be modified by replacing $n^{-1}\log|P_n(z)|$ by $n^{-1}\log|\alpha_n P_n(z)|$, α_n proper constants. The integral above is replaced by a certain real function $\phi(z)$ for which

$$\phi(z) - \int_D \log r \, d\psi(e),$$

D an arbitrary bounded domain, is harmonic in D. All these results are illustrated by various examples and applied in particular to the series of type Σ. (E) Let us consider the situation D and let $\overline{\lim}\,|\alpha_n^{-1}a_n|^{1/n} = \lambda$. Then Σ is convergent or divergent according as $\phi(z) + \log\lambda < 0$ or > 0, respectively; in the latter case again, an exceptional subset of E' might exist which can be covered by a set of circles, the sum of the αth powers of whose radii is less than ϵ. Another more precise theorem is also obtained. (F) An inversion of D is possible in the following sense. The regular distribution of the zeros follows from the existence of a function $\phi(z)$ such that $\overline{\lim}\, n_\nu^{-1}\log|\alpha_{n_\nu}P_{n_\nu}(z)|$ holds almost everywhere for an arbitrary subsequence $\{n_\nu\}$.

G. Szegö (Stanford University, Calif.).

Ibraguimoff, I. Sur quelques systèmes complets de fonctions analytiques. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1939**, 553–568 (1939). (Russian. French summary) [MF 2096]

A set $\{u_n(z)\}$ of analytic functions, regular in $|z|<R$, is complete in $|z|<\rho$, $\rho \leqq R$, if every function $f(z)$, regular in

$|z|<\rho$, is the limit of a sequence of linear combinations of the $u_n(z)$, uniformly in any circle $|z|\leqq\rho-\delta$, $\delta>0$. The author considers the following sets $\{u_n\}$. (I) $z^n e^{\alpha_n z}$ $(n=0, 1, 2, \cdots; |\alpha_n|\leqq 1)$. (II) The same, with $\alpha_n\to 0$. (III) $z^n \exp(ze^{in\alpha})$ $(n=0, 1, 2, \cdots; 0\leqq\alpha<2\pi)$. (IV) $z^n\varphi^{(n)}(z)$ $(n=0, 1, 2, \cdots)$, where $\varphi(z)$ is regular in $|z|<R$ and, for all n, $\varphi^{(n)}(0)\neq 0$. The system (I) is complete at least in the circle $|z|<\log 2$. (II) is complete in any circle. (III) is complete in $|z|<\lambda$, where λ is the absolute value of the zero of smallest absolute value of the function

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\exp\left(-i\alpha\frac{n(n-1)}{2}\right),$$

and not complete in any larger circle. (IV) is complete in $|z|<R$; the condition $\varphi^{(n)}(0)\neq 0$ is necessary. By means of the Borel transform, results on the systems (I), (II), (III) are transformed into results on the convergence of Abel series representing entire functions of exponential type, which yield uniqueness theorems for such functions; for example, associated with (I) is Takenaka's theorem that a function $f(z)$ of exponential type less than log 2, such that $f^{(n)}(\alpha_n)=0$ $(n=0, 1, 2, \cdots; |\alpha_n|\leqq 1)$, vanishes identically [see J. M. Whittaker, Interpolatory Function Theory, 1935, p. 44]. The completeness of (I) for $|z|<1/e$ was established by Takenaka [Proc. Phys.-Math. Soc. Japan (3) **13**, 111–132 (1931)]; for $|z|<\log 2$, completeness and somewhat more have been established independently by the reviewer [Proc. Nat. Acad. Sci. U. S. A. **26**, 139–143 (1940); cf. these Rev. **1**, 229] by a method quite different from that of the author. The author's method for (I) depends on an estimate for the quantities

$$(1)\qquad C_n(z)=\int_{\alpha_0}^{z}\int_{\alpha_1}^{t_1}\cdots\int_{\alpha_{n-1}}^{t_{n-1}}dt_1dt_2\cdots dt_n,$$

namely,

$$|C_n(z)|<A(\log 2)^{-n}2^{|z|},$$

where A is independent of n and z. This he obtains in an elegant manner by use of the recurrence relation expressing $C_n(z)$ in terms of the $C_k(z)$ $(k<n)$, obtained by carrying out the integrations in (1). He is then able to show that

$$e^{zw}=\sum_{n=0}^{\infty}z^n e^{\alpha_n z}C_n(w)$$

for $|z|<\log 2$, and hence, by successive differentiations, that the powers z^k can be expanded in terms of the functions (I). He proceeds similarly with the functions (III). The functions (IV) are treated by showing that $z^n\varphi^{(n)}(0)$ can be expanded in terms of them for every n.

R. P. Boas, Jr. (Durham, N. C.).

Temliakow, A. Über harmonische Funktionen von drei Veränderlichen mit einer meromorphen zugehörigen Funktion. Rec. Math. [Mat. Sbornik] N.S. **5** (**47**), 487–495 (1939). (German. Russian summary) [MF 1339]

The author investigates the properties of the harmonic function $F(x, y, z)$ using its representation as an integral

$$(*)\qquad (2\pi)^{-1}\int_{-\pi}^{\pi}f(e^{it}, u)dt,\quad u=x+iy\cos t+iz\sin t,$$

valid in the neighborhood of every regular point (x^0, y^0, z^0). First an expression for f as a definite integral is obtained, the expression being different from the one given previously by Bergmann [Math. Ann. **99**, 629–659 (1928)]. The author next considers f to be meromorphic. Then if x, y, z vary, the integral (*) may have jumps, and, for different regions G_k^3 in the xyz-space, (*) represents different harmonic functions $F_k(x, y, z)$. He shows that, in going from one region G_k^3 to another, the jumps of (*) are algebraic functions of x, y and z, and are given. The author uses a technique similar to that used by Bergmann for the case when f is rational [loc. cit.]. Some applications are made of these results. In particular, he shows that, in the case of harmonic functions depending on x, and $r=(y^2+z^2)^{\frac{1}{2}}$, the position and nature of the singularities of F can be obtained in terms of the coefficients of the expansion of F.

A. Gelbart.

Whitmore, William F. Convergence theorems for functions of two complex variables. Amer. J. Math. **62**, 687–696 (1940). [MF 2466]

The author proves the following theorem. Given a function $f(z_1, z_2)$ of two complex variables $z_1=x_1+iy_1$, $z_2=x_2+iy_2$ defined and regular in the closed quarter-space $y_1\geqq 0$, $y_2\geqq 0$, with $|f(z_1, z_2)|<\epsilon(0<\epsilon<1)$ for $\{z_1, z_2\}\varepsilon E[y_1=y_2=0, \delta(x_1+a)+x_2^2<0]$, where δ and a are arbitrary positive constants, then $|f(z_1, z_2)|<\epsilon^\mu$ for $\{z_1, z_2\}\varepsilon E[\arg(\delta(z_1+a)+z_2^2)>\mu\pi,\ 1>\mu>0]$. The proof is based on a theorem due to Bergmann [Math. Ann. **109**, 324–348 (1934)] which in its turn is a generalization of a theorem of Ostrowski for the one variable case. The proof consists in finding a double-harmonic function having certain properties which enable one to set up an appropriate double harmonic measure. Mapping the quarter-space onto the bicylinder $|z_1|\leqq r$, $|z_2|\leqq 1$, the author interprets his result for the bicylinder. This result is applied to a more general class of domains, called M-domains. (Domains of this sort have been considered by Bergmann and are a subclass of domains with distinguished surfaces.) The author shows that an M-domain contains a (small) bicylinder and then by means of certain comparisons between the M-domain and the bicylinder he derives a theorem for M-domains similar to those for the quarter-space and the bicylinder.

W. T. Martin.

Lelong, Pierre. Sur l'ordre d'une fonction entière de deux variables. C. R. Acad. Sci. Paris **210**, 470–472 (1940). [MF 2012]

Results of the author concerning the growth of an entire function $F(x, y)$ in a given direction $y=tx$ are noted. Let $F(x, tx)=\Phi(x, t)$ and $M(r, t)$ denote the maximum of $|\Phi|$ on $|x|=r$. The order of $F(x, y)$ for given t is defined by

$$\rho(t)=\overline{\lim}\log_r M(r, t)/\log r.$$

Then $\rho(t)$ has a constant value ρ_0, zero, finite or infinite in every bounded domain of the plane, except perhaps on a set of points of zero capacity, where $\rho(t)<\rho_0$. If a is an exceptional value in the sense of Picard for the entire function $F(x, y)$ on a set E of directions, then either E is a bundle of zero capacity, or a is an exceptional value for every direction and $F(x, y)$ never takes on the value a. If $F(x, y)$ is a meromorphic function, the set of directions $y=tx$, where $F(x, tx)$ is an entire function of x, form a bundle of zero capacity.

M. S. Robertson.

Fuchs, B. A. On pseudo-conformal mappings of a region in itself with a fix-point on the boundary. Doklady Akad. Nauk SSSR (N.S.) **23**, 865–867 (1939). (Russian) [MF 1964]

In order to study pseudo-conformal transformations, Bergmann associated with a domain B the kernel $K=K_B(z, t)$ of a complete set of orthogonal functions, and the invariant

metric $(ds)^2 = T_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$, $T_{\alpha\bar{\beta}} = \partial^2 \log K / \partial z^\alpha \partial \bar{z}^\beta$. If $C(B)$ is the class of domains obtained from B by a pseudo-conformal transformation normed with respect to t, $t \in B$, he chooses from $C(B)$ a "representative domain" $R(B, t)$ which is obtained from B by $w^\nu(z, \bar{t}) = T^{\nu\tau}(t)\partial[\log K(z, \bar{t})/K(t, \bar{t})]/\partial \bar{t}^\tau$. If B admits a pseudo-conformal transformation V into itself with fixed point t, then there corresponds to V a linear transformation of $R(B, t)$ into itself. [Bergmann, J. Reine Angew. Math. **169**, 1–42 (1932); **172**, 89–128 (1934); Math. Ann. **102**, 430–446 (1930)]. In addition to the above and with the aid of certain of the author's previous results [Fuchs, Rec. Math. [Mat. Sbornik] N.S. **2** (**44**), 567–597 (1937); Bull. Inst. Math. Mec. Tomsk **1**, 281–285 (1927)], the author deals with the case when the point t is a boundary point. Let $\{t_i\}$ be a set of points of B converging to t, and let $\delta z^\nu = \xi^\nu(z^1, z^2)\delta p$ be an infinitesimal transformation that leaves $(ds_B)^2$ invariant. Let $\lim_{\{t_i\} \to t} \xi^\nu(z^1, z^2) = 0$. If certain limit relations exist for $\xi^\nu(t_i)$, then

$$\lim_{t_i \to t} (T^{\nu\tau} K)^{-\beta_\nu} K^{-\alpha} U^\nu(z, \bar{t}_i) = V^\nu(z)$$

exists, β_ν and α being suitable constants. $V^\nu = V^\nu(z)$ transforms B into a domain $R[B, t]$ such that to the transformations of the group of analytic movements, obtained from the infinitesimal transformations $\partial z^\nu = \xi^\nu \delta p$, there corresponds in $R[B, t]$ a linear and homogeneous transformation of $R[B, t]$ into itself. Using Bergmann's results [loc. cit.] on the behavior of K on the boundary, the author further deals with the particular case when t is a limit point of the second or third order. *A. Gelbart* (Cambridge, Mass.).

Cartan, Henri. Sur les matrices holomorphes de n variables complexes. J. Math. Pures Appl. **19**, 1–26 (1940). [MF 1988]

Verfasser betrachtet im Raume von n komplexen Veränderlichen quadratische, über einem gegebenen Bereiche Δ analytische und umkehrbare Matrizen, das heisst, Matrizen, deren Elemente im abgeschlossenen Bereiche Δ analytische Funktionen sind und deren Determinante in keinem Punkte von Δ verschwindet. Zunächst wird ein Satz von Cousin [vgl. auch H. Cartan, C. R. Acad. Sci. Paris **199**, 925–927 (1934)] aus der Theorie der analytischen Funktionen mehrerer Veränderlichen auf solche Matrizen übertragen. Es gilt (Satz I): Im Raume von n komplexen Veränderlichen seien zwei kompakte (das heisst, beschränkte und abgeschlossene) Polyzylinder Δ' und Δ'' gegeben, deren Projektionen auf die Koordinatenebenen alle bis auf eine Ausnahme zusammenfallen und deren Durchschnitt $\Delta = \Delta' \cap \Delta''$ einfachzusammenhängend ist. Dann lässt sich jede über Δ analytische, umkehrbare Matrix A in der Form $A = A'^{-1}A''$ darstellen, wobei A' eine über Δ', A'' eine über Δ'' analytische, umkehrbare Matrix ist. Es werden dann mit Hilfe dieses Satzes gewisse Eigenschaften der Ideale analytischer Funktionen untersucht. Hierzu folgendes: Die durch eine über einem Bereiche Δ analytische, p-reihige Matrix A definierte Transformation, angewandt auf ein System von p in Δ analytischen Funktionen $(f_1, \cdots, f_p)$ führt dieses wieder in ein System von in Δ analytischen Funktionen $(g_1, \cdots, g_p)$ über; Bezeichnung: $(g) = A(f)$. Es seien nun Δ' und Δ'' Bereiche, die den Bedingungen des Satzes I genügen. Ist dann $\mathfrak{J}'$ ein Ideal über Δ' von im abgeschlossenen Δ' analytischen Funktionen mit der Basis $(f_1', \cdots, f_p')$, $\mathfrak{J}''$ ein Ideal über Δ'' mit der in Δ'' analytischen Basis $(f_1'', \cdots, f_p'')$ und gibt es ferner eine über $\Delta = \Delta' \cap \Delta''$ analytische, umkehrbare Matrix A, sodass $(f') = A(f'')$, so folgt fast unmittelbar aus Satz I, dass ein System von p Funktionen existiert, die in dem Vereinigungsbereich von Δ' und Δ'' analytisch sind und gleichzeitig als Basis von $\mathfrak{J}'$ über Δ' und von $\mathfrak{J}''$ über Δ'' dienen. In dem gleichen Zusammenhange wird der Satz II bewiesen: Die Bereiche Δ' und Δ'' mögen den Bedingungen des Satzes I genügen; $\mathfrak{J}'$ sei ein Ideal von analytischen Funktionen über Δ', $\mathfrak{J}''$ ein solches über Δ'', beide mit endlicher Basis. Damit die Ideale $\mathfrak{J}'$ und $\mathfrak{J}''$ dieselbe analytische Basis über dem Vereinigungsbereich von Δ' und Δ'' zulassen, ist notwendig und hinreichend, dass $\mathfrak{J}'$ und $\mathfrak{J}''$ über dem Durchschnitt $\Delta = \Delta' \cap \Delta''$ das gleiche Ideal erzeugen. Hierbei besteht das von dem Ideal $\mathfrak{J}'$ (bzw. $\mathfrak{J}''$) über Δ erzeugte Ideal aus den endlichen Kombinationen von Funktionen aus $\mathfrak{J}'$ (bzw. $\mathfrak{J}''$) mit im abgeschlossenen Δ analytischen Koeffizienten. Verfasser behandelt zuletzt folgendes Problem: $\mathfrak{J}$ sei ein Ideal über Δ und besitze zwei Basen mit der gleichen Anzahl p von Basisfunktionen. Gibt es dann eine über Δ analytische, umkehrbare, p-reihige Matrix, welche die eine Basis in die andere überführt? Verfasser beweist in zwei Spezialfällen die Möglichkeit einer solchen Transformation, weist aber zugleich nach, dass die Frage im Allgemeinfall zu verneinen ist. Die beiden Spezialfälle sind: (1) Haben die Funktionen eines Ideals über einem kompakten Polyzylinder Δ dort keine gemeinsamen Nullstellen, so sind stets zwei Basen, welche die gleiche Anzahl p von Basisfunktionen besitzen, durch eine über Δ analytische, umkehrbare, p-reihige Matrix ineinander überführbar. Ein solches Ideal ist insbesondere notwendig mit dem Einheitsideal identisch. (2) Zwei Basen (f_1, f_2) und (g_1, g_2) eines Ideals $\mathfrak{J}$ über dem kompakten Polyzylinder Δ sind immer dann durch eine über Δ analytische, umkehrbare Matrix ineinander überführbar, falls die von den gemeinsamen Nullstellen der Funktionen von $\mathfrak{J}$ gebildete $(n-2)$-dimensionale Mannigfaltigkeit einfachzusammenhängend ist. *P. Thullen* (Quito).

Wagner, R. W. An extension of analytic functions to matrices. Amer. J. Math. **62**, 380–390 (1940). [MF 1772]

Analytic functions of matrices are obtained as power series in matrices, together with all their analytic continuations in each of the variables. Several theorems on singularities are proved. Finally it is shown that the solutions of polynomial matrix equations studied by Roth and Franklin may be regarded as branches of an analytic function of the kind here defined. *P. Franklin* (Cambridge, Mass.).

Beckenbach, E. F. Space analogs of function-theoretic results. Amer. Math. Monthly **47**, 199–211 (1940). [MF 1925]

The fact that two functions $x(u, v)$, $y(u, v)$ are conjugate harmonic can be expressed in terms of the Cauchy-Riemann equations, in terms of the theorem of Morera and in several other ways. These various characterizations of pairs of conjugate harmonic functions suggest generalizations to triples of functions $x(u, v)$, $y(u, v)$, $z(u, v)$ which are thought of as the coordinate functions of a surface S. In this manner one is led to a variety of results in the differential geometry of surfaces which may be considered as partial generalizations of fundamental facts in the theory of analytic functions of a complex variable. The present paper gives a comprehensive survey of the relevant literature which includes contributions due to the author himself. *T. Radó.*

Differential Equations

Rodabaugh, Louis D. The partial differential equation $\partial z/\partial x+f(x, y)\partial z/\partial y=0$. Duke Math. J. 6, 362–374 (1940). [MF 2321]

Let g be a bounded, open, simply connected plane region. Let the equation

$$(1) \qquad \frac{\partial z}{\partial x}+f(x, y)\frac{\partial z}{\partial y}=0$$

be called regular if there exists a function $\psi(x, y)$ of class C' with respect to x and y in g, which is a solution of (1) and such that also in g the inequality $\psi_y(x, y)>0$ holds. The author proves that the continuity of $f(x, y)$ and $f_y(x, y)$ in the closure of g is a sufficient condition for (1) to be regular. The corresponding problem for multiply connected regions is also handled. *F. G. Dressel* (Durham, N. C.).

Saltykow, N. Étude sur les intégrales de S. Lie des équations aux dérivées partielles du premier ordre à une fonction inconnue. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. 5, 121–137 (1939). [MF 1919]

A critical review of Cauchy's and Lie's methods of solution of first order partial differential equations. Although Lie's definition of an integral is more general than the classical one, it is proved that any Lie integral is obtainable from the complete integral. A complete integral of Lie of class q for the equation $p_1+H(x_i, z, p_2, \cdots, p_n)=0$, $i=1, \cdots, n$, is defined as

$$z=\phi(x_1, \cdots, x_{n-q}, C_1, \cdots, C_n),$$
$$x_{n-q+i}=\phi_i(x_1, \cdots, x_{n-q}, C_1, \cdots, C_n).$$

It is then proved that a nonlinear equation in two variables does not admit a complete Lie integral and the result is extended to a compatible system of m equations in n variables. If these equations are nonlinear, the system does not admit a complete Lie integral of class $n-m$.

M. S. Knebelman (Pullman, Wash.).

Saltykow, N. Forme canonique des groupes fonctionnels. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. 5, 75–87 (1939). [MF 1918]

A differential equation $F(x_i, p_i)=0$, $i=1, \cdots, n$; $p_i=\partial z/\partial x_i$, being given, Lie's method of solution yields the theorems: (1) If the equation of the characteristics admit r independent integrals in involution, there exists a system of $2n-r$ differential invariants by means of which the equation $F=0$ can be reduced to one in $n-r$ independent variables. (2) If the equations of the characteristics admit a function group of r distinct integrals having μ singular (distinguées) functions, then there exists a system of $2(n-\rho)-\mu$ differential invariants ($r-\mu=2\rho$), by means of which the equation $F=0$ can be reduced to one in $n-\rho-\mu$ independent variables. There is also a brief application of these theorems to the problem of three bodies.

M. S. Knebelman (Pullman, Wash.).

Chow, Wei-Liang. Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung. Math. Ann. 117, 98–105 (1939). [MF 1386]

Carathéodory [Math. Ann. 67, 355–386 (1909), in particular p. 369] has proved the following theorem. If a Pfaffian equation at each point P has the property that, in every neighborhood of P, points exist which cannot be reached from P by an integral curve of the equation, then the equation is completely integrable. The present paper extends this result as follows. If a Pfaffian system A_r at each point P has the property that, in every neighborhood of P, the points which can be reached from P by an integral curve of the system form a manifold of at most $n-r+m$ dimensions, then the system has at most index m. In the proof the Pfaffian system A_r is replaced by the "associated" system B_{n-r} of linear homogeneous partial differential equations of first order. The index of a Pfaffian system is the number of equations which must be added to the associated system to make it complete. The integral curves of A_r used in the proof are compounded from the characteristics of the various equations of B_{n-r}. *E. W. Titt* (Hyattsville, Md.).

Lukchin, V. S. Invariabilité du nombre de fonctions arbitraires dépendant du plus grand nombre d'arguments dans le calcul inverse de la dérivation. C. R. (Doklady) Acad. Sci. URSS (N.S.) 24, 643–647 (1939). [MF 2053]

In Janet's [Leçons sur les systèmes d'équations aux dérivées partielles, Paris, 1929] treatment of the system of partial differential equations

$$(1) \qquad \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}u=f_{\alpha_1\cdots\alpha_n}(x_1, \cdots, x_n),$$

the number of arbitrary functions in the initial determination [cf. J. M. Thomas, Differential Systems, New York, 1937, p. 57] is not an invariant of the system but depends on the ordering of the independent variables $x_1, \cdots, x_n$. The same statement holds for J. M. Thomas' treatment by the method of relative complete sets. In the present paper, however, the author shows that the number of functions in the initial determination having the maximum number of independent variables as arguments is an invariant of the system (1). *F. G. Dressel* (Durham, N. C.).

Kosambi, D. D. The tensor analysis of partial differential equations. J. Indian Math. Soc. 3, 249–253 (1939). [MF 1020]

Following the work of J. Douglas, E. Bortolotti and others this paper considers the system of partial differential equations

$$(1) \qquad \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}+H^i_{\alpha\beta}(u^\gamma, x^j, p_\gamma{}^r)=0; \quad p_\alpha{}^i=\frac{\partial x^i}{\partial u^\alpha},$$

and similar systems of higher order from the differential geometer's point of view. The main steps of the present treatment are outlined in what follows. First, the system (1) is written in invariantive form under the group of transformations

$$\bar{x}^i=\bar{x}^i(x^1, \cdots, x^n); \quad \bar{u}^\alpha=\bar{u}^\alpha(u^1, \cdots, u^m).$$

The next step in the process is to write the equations of variation of the system (1) (in the sense of the calculus of variations) in invariantive form. The contractions of certain tensors thus encountered are equated to zero in order to determine the connection in terms of the $H^i_{\alpha\beta}$ and their derivatives. In all, three types of differentiation which generate invariants are introduced: first, $\partial/\partial p_\alpha{}^i$; second, a type of covariant differentiation with respect to u^α; third, an alternation of these two. *E. W. Titt* (Hyattsville, Md.).

Buzano, Piero. Bericht über die projektive Geometrie der partiellen Differentialgleichungen. Abh. Math. Sem. Hansischen Univ. 13, 257–272 (1940). [MF 2384]

This paper is a brief outline of matter, methods and

results of recent Italian investigations of partial differential equations from a geometrical viewpoint. It concerns systems of linear equations whose integral depends on arbitrary functions, in particular, systems of "Laplace equations" (second order) in two or more independent variables. Bompiani rendered projective language applicable to this work by dealing with configurations in the higher osculating spaces of the manifolds represented by the given equation system. His quasi-asymptotic lines are particularly useful. The principal developments discussed are: (i) early investigations of surfaces satisfying Laplace equations (Segre), (ii) characterization of surfaces by quasi-asymptotic lines (Bompiani, Villa), (iii) manifolds ($n>2$) satisfying sufficiently many Laplace equations (Terracini), (iv) manifolds satisfying two or three Laplace equations (Buzano).

J. L. Vanderslice (Bethlehem, Pa.).

Tweedie, M. C. K. The solution of a certain class of differential equations. Math. Gaz. **24**, 25–29 (1940). [MF 1457]

This paper considers the second order partial differential equation

$$\alpha u_{xx}+\beta u_{xy}+\gamma u_{yy}+\delta u_x+\epsilon u_y+\eta u=0,$$

where the coefficients are functions of x alone. We look for a solution of the form

$$u=\sum_{i=0}^{\infty}\phi_i(x)\frac{d^i}{dy^i}f(y),$$

where f is arbitrary. We arrive at an infinite system of second order ordinary differential equations which are recurrence formulae for the ϕ_i. Convergence of the solution is not considered. A variation of the method for the case of ordinary equations is mentioned. *E. W. Titt.*

Doetsch, Gustav. Ein Zusammenhang zwischen Randwertproblemen verschiedenen Typs. Math. Z. **46**, 315–328 (1940). [MF 2401]

Simple integral combinations of solutions of the elliptic, hyperbolic and parabolic partial differential equations $U_{xx}+U_{yy}=0$, $V_{xx}-V_{yy}=0$, $W_{xx}-W_y=0$ are first obtained. For example, when $W_1(x, y)$ and $W_2(x, y)$ are two solutions of the parabolic equation, the integral

$$U(x, y)=\int_0^{\infty}W_1(x, t)W_2(y, t)dt$$

is a solution of the elliptic equation. Conditions on the regularity of the integral and the limits of W_1 and W_2 as $t\to 0$ and $t\to\infty$ are involved. Either factor in the integrand, or both factors, may be replaced by a solution of either the elliptic or hyperbolic equation to form a new integral which is again a solution of one of the three equations. A table is given to show which equation is satisfied by each integral combination. The correspondence between boundary conditions of the component and resultant solutions is examined in the case of certain singular solutions (solutions which vanish on the entire boundary) of the parabolic equation. This leads to a class of singular solutions of the elliptic equation. *R. V. Churchill* (Ann Arbor, Mich.).

Mushelišvili, N. I. On the solution of Dirichlet's problem for the plane. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] **1**, 99–106 (1940). (Russian) [MF 2018]

In this paper a new variant of the reduction of the problem of Dirichlet to the solution of Fredholm's integral equation is given. Let S^2 be a domain bounded by $(n+1)$ closed curves L_k^1 with continuously varying curvature and let $L^1=\sum_{k=1}^{n}L_k^1$. Let $\mu(t)$ be a real function defined on L^1. The author shows that, by Plemelj's theorem [J. Plemelj, Potentialtheoretische Untersuchungen, Leipzig, 1911], the first boundary problem (namely, to find the harmonic function equal to $f+a_k$ on L_k^1, where f is a given function, $a_0=0$, a_k, $k=1, \cdots, n$, constants to be determined) can be reduced to the solution of the equation

$$\mu(\tau)+\pi^{-1}\int_{L^1}\mu(t)d\theta(t, \tau)=f(\tau)+a_k, \qquad \tau\epsilon L_k^1.$$

Here $\theta(t, \tau)$ is the angle formed by the vector $\overrightarrow{\tau t}$ and the positive x-axis. The author shows, furthermore, that instead of the equation considered one can use the equation

$$\mu(\tau)+\pi^{-1}\int_{L^1}\mu(t)d\theta(t, \tau)-\int_{L^1}k(t, \tau)\mu(t)ds=f,$$

where $k(t, \tau)$ is $\rho_k(t)$ for $\tau\epsilon L_k^1$, $t\epsilon L_k^1$ and is 0 in all other cases. $\rho_k(t)$ is an arbitrary continuous function for which

$$\int_{L_k^1}\rho_k(t)ds\neq 0,$$

where s is the arc corresponding to the point t.

S. Bergmann (Cambridge, Mass.).

Mushelišvili, N. I. Remarks concerning fundamental boundary problems in potential theory. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] **1**, 169–170 (1940). (Russian) [MF 2020]

In the paper reviewed above the author considered the equation

$$\mu(\tau)+\lambda\pi^{-1}\int_{L^1}\mu(t)d\theta(t, \tau)-\lambda\int_{L^1}k(t, \tau)\mu(t)ds=0$$

in connection with the problem of Dirichlet. He proves here that the fundamental values λ of this equation are real and that, furthermore, the interval $-1\leqq\lambda\leqq 1$ does not contain fundamental values if

$$0<-\int_{L_k^1}\rho_k(t)ds<2, \qquad k=1, 2, \cdots, n.$$

S. Bergmann (Cambridge, Mass.).

Cimmino, Gianfranco. Equazione di Poisson e problema generalizzato di Dirichlet. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 322–329 (1940). [MF 2168]

The author states theorems, the proofs of which are to be published later, relative to Poisson's equation $\Delta u=f$ in n dimensions. These theorems deal with the unicity and existence of solutions among a class of solutions u which assume "in the mean of order p" ($p>1$), over a set of smooth uniformly approximating boundaries, the assigned frontier values ϕ, the case $p=1$ having been previously studied under more restrictive conditions on the boundary. By means of a spherical mean value identity, corresponding in three dimensions to an iterated integration over a sphere of the equation $\Delta u=f$, the author establishes a convergence theorem on sequences of solutions $u^{(\nu)}$ corresponding to sequences of assigned functions $\phi^{(\nu)}$, the convergences being in the mean of orders $q'>1$ and $p>1$. He discusses the correspondence between certain complete abstract spaces based on the functions ϕ, f and u, Δu, respectively.

G. C. Evans (Berkeley, Calif.).

Simonoff, N. Über die erste Randwertaufgabe der nichtlinearen elliptischen Gleichung. Bull. Math. Univ. Moscou 2, no. 1, 18 pp. (1939). [MF 1997]

The author considers a nonlinear elliptic partial differential equation of the form

$$(1) \qquad F(x, y, u, p, q, r, s, t) = 0,$$

$$p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y}, \quad r = \frac{\partial^2 u}{\partial x^2}, \quad s = \frac{\partial^2 u}{\partial x \partial y}, \quad t = \frac{\partial^2 u}{\partial y^2}.$$

He assumes that F is defined and of class C' for $|x|, |y| < B$ and all u, p, q, r, s, t, B being some positive constant. In the above domain, it is assumed that there exist positive constants k_1, k_2, k_3 such that

$$4F_rF_t - F_s^2 \geqq k_1,$$

$$F_r > 0; \ |F_u|, |F_p|, |F_q|, |F_r|, |F_s|, |F_t| \leqq k_2; \ F_u \leqq -k_3.$$

A region G may be said to be normal if it is bounded by a simple closed curve R such that if P is any point of R, there exists a closed circle K_P such that $K_P \cdot (G+R) = P$. The author shows that if G is any normal region lying in the square $|x|, |y| < B$, then there exists a unique solution of (1) which takes on any given continuous boundary values on R. If F_u does not satisfy the above condition, the solution exists for any sufficiently small normal region. These results are all generalized to a similar type of equation in n independent variables. The results are proved by an extension of the method of Perron for the solution of the Dirichlet problem. *C. B. Morrey, Jr.* (Berkeley, Calif.).

Solovieff, P. V. Résolution des équations du type elliptique et du type parabolique pour les petits domaines. Rec. Math. [Mat. Sbornik] N.S. 5 (47), 473–486 (1939). (Russian. French summary) [MF 1338]

The author investigates the existence of the solution of $\sum_{i=1}^{n} u_{x_i x_i} = f(x_i; u, u_{x_i})$, where $u_{x_i} = \partial u / \partial x_i$, which vanishes on the boundary Ω of the hypersphere H, $\sum x_i^2 < \sigma^2$. It is known that u satisfies a system of integral equations

$$u_{x_p}(y_i) = -\sigma^{-1} \int_{\Omega} f(x_i; u, u_{x_i}) G_{x_p}(x_i, y_i) du,$$

$G(x_i, y_i)$ being the Green's function for H. The above system of integral equations defines a transformation in the function space. Supposing that

$$|f(x_i; y_p) - f(x_i; y_p')| \leqq \sum_{p=1}^{n+1} A_p |y_p - y_p'|^2$$

and $\int_{\Omega} |f(x_i; 0)|^q d\omega = c < \infty$, the author proves that the right hand members of the integral equations above satisfy certain inequalities. Following the method used by Niemytzski [Rec. Math. [Mat. Sbornik] N.S. 1 (43), 485–502 (1936)], and by the theorem of Cacciopola-Banach on the existence of fixed points for the transformations considered, he proves: there exists one and only one solution u, $u \in L_{n+1}^q$, vanishing on Ω, if σ is sufficiently small, L_{n+1}^q being the space of functions integrable with q derivatives. An analogous result is proved in the case of $\sum_{i=1}^{n} u_{x_i x_i} = f_1(x_i; u)$ for a sufficiently small domain D, and in the case of $u_{xx} - \phi(x, y, u) u_y = 0$, if $|\phi(x, y, u) - \phi(x, y, u')| \leqq A |u - u'|$, $\iint_D |\phi(x, y, 0)|^q dx dy = c < \infty$. In particular the author considers the case when D is a rectangle or a trapezoid. *S. Bergmann.*

Karimov, Dj. Kh. Sur les solutions périodiques des équations différentielles non-linéaires du type parabolique. C. R. (Doklady) Acad. Sci. URSS (N.S.) 25, 3–6 (1939). [MF 2062]

For the non-linear differential equation

$$\frac{\partial z}{\partial t} - a^2 \frac{\partial^2 z}{\partial x^2} = \Phi(x, t) + uf(z), \qquad 0 \leqq t; \ x \leqq 0,$$

where $\Phi(x, t+1) = \Phi(x, t) = \sum_{n=1}^{\infty} \Phi_n(t) \sin n\pi x$, the following periodic boundary value problem is handled by the method of successive approximations: $z(0, t) = z(1, t) = 0$; $z(x, 0) = z(x, 1)$. *F. G. Dressel* (Durham, N. C.).

Solovieff, P. V. Sur les solutions périodiques de certaines équations non-linéaires du quatrième ordre. C. R. (Doklady) Acad. Sci. URSS (N.S.) 25, 731–734 (1939). [MF 2127]

The same type of periodic boundary value problem handled by Karimov [see above review] is here handled for the equation

$$\frac{\partial^2 z}{\partial t^2} + \frac{(2K+1)^2}{\pi^2} \frac{\partial^4 z}{\partial x^4} = \sum_{n=0}^{\infty} \Phi_{2n+1}(t) \sin (2n+1)\pi x + uf(z),$$

k an integer.

F. G. Dressel (Durham, N. C.).

Ornstein, L. S. and Milatz, J. M. W. Accidental deviations in the conduction of heat. Physica 6, 1139–1145 (1939). [MF 866]

To the differential equation of the conduction of heat a term is added to describe random fluctuations. The procedure is analogous to that in the theory of Brownian movement. *W. Feller* (Providence, R. I.).

***Baker, Bevan B. and Copson, E. T. The Mathematical Theory of Huygens' Principle.** Oxford University Press, New York, 1939. vii+155 pp. $4.25.

Huygens' geometrical construction, with its restriction that only one sheet of the envelope of the spherical wavelets is to be considered, is first justified in Chapter I by Poisson's analytical solution of the equation of wave-motions. A discussion is then given of the ideas of Fresnel and of the formula of Helmholtz which expresses these ideas in analytical form and gives the principle of Huygens for periodic processes. The diffraction formulae of Fresnel and Stokes are then obtained.

Kirchhoff's famous formula is first derived from the formula of Helmholtz and then proved directly. The formula is interpreted physically and the question of the uniqueness of the solution discussed. It is pointed out that to extend the theorem of Kirchhoff to the space outside a closed surface it is necessary to prescribe the asymptotic behaviour of the wave-functions under consideration. The peculiarities of wave-propagation in two dimensions are next indicated and Weber's analogue of Helmholtz's theorem is given. The analogue of Kirchhoff's formula, due to Volterra, is derived and an interesting account is given of a method, devised by Marcel Riesz and based on the theory of fractional integration, which provides a powerful method of solving initial value problems for equations like the wave equation.

The rest of the book is devoted chiefly to the problem of diffraction. Part of Chapter II includes a useful discussion of an important type of definite integral which occurs in the analysis of diffraction problems. Diffraction by a black screen is discussed in some detail. In Chapter III Huygens'

principle is formulated for electromagnetic waves and Tedone's proof is given for some formulae which are associated with the names of Larmor and Tedone. Chapter IV contains a good account of Sommerfeld's theory of diffraction.

H. Bateman (Pasadena., Calif).

Mindlin, J. A. Propagation of waves in two dimensions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 280–284 (1939). [MF 2067]

Mindlin, J. A. Solution of Cauchy-Dirichlet's external problem for a wave equation in the case of a circle. C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 285–288 (1939). [MF 2068]

Mindlin, J. A. Propagation of elastic waves in two dimensions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 289–292 (1939). [MF 2069]

In the first paper it is shown that the solutions of the wave equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=\frac{1}{a^2}\frac{\partial^2 U}{\partial t^2} \tag{1}$$

admit representation by the Fourier series

$$U=\sum e^{in\theta}\int_0^\infty [A_n^{(1)}(at-r\cosh\xi)+A_n^{(2)}(at+r\cosh\xi)]\cosh n\xi\, d\xi$$

in analogy to d'Alembert's solution for the string. The function A_n can be so determined that U satisfies initial conditions for $t=0$. The results are used in the second paper which treats the exterior Cauchy-Dirichlet problem for (1): initial data for $r\geqq 1$ and U prescribed for $r=1$, $t>0$. The function A_n is found explicitly by solving a Volterra integral equation. The same problem for a system of two equations (1) is treated in the third paper.

K. Friedrichs (New York, N. Y.).

Einaudi, Renato. Il problema misto per l'equazione delle onde. Atti Accad. Sci. Torino **74**, 470–480 (1939). [MF 1319]

Let S be a domain of the three dimensional x, y, z space bounded by the surface σ. The author proves uniqueness and existence for the so-called mixed problem concerning the equation $\partial^2 f/\partial t^2=\Delta f$ which consists of finding a solution of this equation satisfying the conditions

$$f(x,y,z,0)=u(x,y,z), \qquad \left(\frac{\partial f}{\partial t}\right)_{t=0}=v(x,y,z),$$

and

$$f(x,y,z,t)=w(x,y,z,t) \qquad \text{on } \sigma.$$

The assumptions concerning the given functions u, v, w are as follows: u and v and their derivatives up to the fourth order are continuous; w and its derivatives with respect to t up to the fifth order are continuous; on σ we have

$$u(x,y,z)=w(x,y,z,0), \quad v(x,y,z)=\left(\frac{\partial w}{\partial t}\right)_{t=0},$$

$$\Delta u=\left(\frac{\partial^2 w}{\partial t^2}\right)_{t=0}, \qquad \Delta v=\left(\frac{\partial^2 w}{\partial t^3}\right)_{t=0}.$$

The method used is the one of transforming the problem into an integral equation. *E. Rothe* (Oskaloosa, Iowa).

Einaudi, Renato. Metodo risolutivo in un dominio sferico del problema misto per l'equazione delle onde. Atti Accad. Sci. Torino **74**, 481–491 (1939). [MF 1320]

In the special case that the domain S is a sphere the author treats the same problem as in the preceding review and shows how in this case the solution can be calculated by expansion in spherical harmonics with coefficients depending on $r=(x^2+y^2+z^2)^{\frac{1}{2}}$ and the time t. *E. Rothe.*

Einaudi, Renato. Sul moto di una sfera elastica. Atti Accad. Sci. Torino **74**, 591–618 (1939). [MF 1730]

In a paper to appear in the Annali di Matematica [Un problema fondamentale della dinamica dei sistemi elastici] the author treats the problem of determining the motion of an elastic body C if the displacement is given at the boundary and the initial displacement and velocity in the interior of C. The aim of the present paper is to give a simpler solution in the case that C is a sphere. In constructing an expansion for the solution, the author makes use of expansions which he obtained in the paper reviewed above.

E. Rothe (Oskaloosa, Iowa).

TOPOLOGY

Franklin, Philip. The four color problem. Scripta Math. **6**, 149–156, 197–210 (1939). [MF 2478]

Winn, C. E. On the minimum number of polygons in an irreducible map. Amer. J. Math. **62**, 406–416 (1940). [MF 1774]

It is here proved that every map with 35 or fewer regions on a plane or sphere is colorable in four colors. This improves the similar result for 31 obtained by Franklin. The argument makes use of the known reductions of Birkhoff, Errera, Franklin and the author, including some newly developed in the present paper, and centers around a series of inequalities in which the number of pentagons in the map figure prominently. *P. Franklin* (Cambridge, Mass.).

Lebesgue, Henri. Quelques conséquences simples de la formule d'Euler. J. Math. Pures Appl. **19**, 27–43 (1940). [MF 1989]

The polyhedra considered in this paper are of genus zero, with simply-connected faces, each having at least three sides. By a straightforward argument, the author obtains the formula $\sum\sum(s^{-1}+f^{-1}-\frac{1}{2})=2$, where the summation is over all the angles of all the faces, s being the number of edges at a vertex and f the number of sides of a face. Grouping together the angles at each vertex, he deduces that $\sum(1-\frac{1}{2}s+\sum f^{-1})=2$, where the outer summation is over the vertices and the inner over the s faces that surround one vertex. It follows that $1-\frac{1}{2}s+\sum f^{-1}>0$ for at least one of the vertices. Hence every polyhedron has either a trihedral vertex with $\sum f^{-1}>\frac{1}{2}$, or a tetrahedral vertex with $\sum f^{-1}>1$, or a pentahedral vertex with $\sum f^{-1}>\frac{3}{2}$. (In the last case four of the f's must be 3, and the fifth less than 6.) Many similar results are obtained, of which the following is typical. If all the vertices are trihedral and every face has at least five sides (as in most discussions of the four-color problem), there are at least twelve pentagons each adjacent to two other faces having at most six sides, and there are at least twelve pentagons each adjacent to three other faces having at most seven sides. *H. S. M. Coxeter.*

Wylie, S. Duality and intersection in general complexes. Proc. London Math. Soc. (2) **46**, 174–198 (1940). [MF 1502]

This paper shows how the classical theory of duality and intersection for combinatorial manifolds [see Lefschetz, Topology, 1930] can be extended in a direct way to general simplicial complexes to duplicate the accomplishments of cohomology and multiplication [see Čech, Ann. of Math. (2) **37**, 681–697 (1936)] and even to give greater generality in the results. The basic difference between the classical approach and the "co" one is that where the latter dualizes the boundary operator the former dualizes the complexes themselves. The author feels that the classical method is the more geometric and so the more natural in topology. For his dual complexes he takes two geometric realizations: (1) a relative one by embedding in an orientable manifold and (2) an intrinsic one formed in the first derived. He reaches a duality theorem which includes all the known duality theorems for complexes, particularly those of Čech [l.c.] depending on p-regularity. The last part of the paper deals with intersections on a general complex and gives geometric content to the somewhat artificial multiplications of "co"theory. *A. W. Tucker* (Princeton, N. J.).

Sumner, D. B. Orientable manifolds constructed from a solid cube. Trans. Roy. Soc. South Africa **28**, 183–197 (1940). [MF 1995]

By identifying in pairs the faces of a solid cube, various complexes are formed. The author investigates those which are orientable manifolds, giving their Betti and torsion numbers, and generators and relations of their fundamental groups. *H. Whitney* (Cambridge, Mass.).

Elsholz, L. Die Länge einer Mannigfaltigkeit und ihre Eigenschaften. Rec. Math. [Mat. Sbornik] N.S. **5** (47), 565–571 (1939). (Russian. German summary) [MF 1348]

The paper deals with the notion of the length of a manifold introduced by the author and S. Froloff. (The length of a closed set A in a manifold M^n is the greatest integer l such that for any $\epsilon>0$ there exist in M^n $l+1$ cycles different from M^n whose intersection is vacuous and at least one of which is contained in $U(A\epsilon)$ [see Froloff-Elsholz, Rec. Math. [Mat. Sbornik] **42**, 637–643].) The author establishes several simple properties of this notion, among them the relations

$$\log_M A \leqq \mathrm{kat}_M A - 1, \quad \log_M M \geqq R(M),$$

where $\log_M A$ denotes the length of the set A in M, $\mathrm{kat}_M A$ the Lusternik-Schnirelman category of A with respect to M and $R(M)$ the rank of M in the sense of Lusternik-Schnirelman. *W. Hurewicz* (Chapel Hill, N. C.).

Cairns, Stewart S. Triangulated manifolds which are not Brouwer manifolds. Proc. Nat. Acad. Sci. U. S. A. **26**, 359–361 (1940). [MF 2146]

The author announces the construction of a triangulation τ of an n-simplex s ($n=3, 4, \cdots$) such that τ is not homeomorphic to any rectilinear triangulation of s. Let a "star-m-manifold" be a triangulated manifold in which each star of a vertex is an m-cell; a "Brouwer-m-manifold" one in which each star can be mapped into m-space E^m so that each cell of the star is a simplex. Then for $m>3$, there exist star-manifolds which are not Brouwer manifolds. There are triangulated m-spheres ($m\geqq 3$) with no convex polyhedral representation in E^{m+1}. The minimum n for which a complex K has a polyhedral representation in E^n is not, for all K, invariant under subdivisions; neither is Brouwer's definition of a manifold. *H. Whitney* (Cambridge, Mass.).

Hirsch, Guy. Détermination d'un nombre minimum de points fixes pour certaines représentations. Bull. Sci. Math. **64**, 45–55 (1940). [MF 2010]

For a very special class of polyhedra P and a restricted class of continuous mappings $f(P)\subset P$ the existence of at least two fixed points is established. *S. Eilenberg.*

Vickery, C. W. Axioms for Moore spaces and metric spaces. Bull. Amer. Math. Soc. **46**, 560–564 (1940). [MF 2430]

The author gives a set of five axioms in terms of point and region and proves that they constitute necessary and sufficient conditions for a space to be metric and complete. A space is metric if and only if it satisfies the first three and the fifth of the axioms. The first three alone are shown to be equivalent to R. L. Moore's Axiom 1_0 and the first four alone to Moore's Axiom 1. *G. T. Whyburn.*

Bebutoff, M. and Schneider, V. On a denumerable topological space. Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 157–160 (1939). (Russian. German summary) [MF 2110]

Example of a countable space which satisfies the first axiom of countability at no point. *J. W. Tukey.*

Fantappiè, Luigi. Lo spazio funzionale analitico come spazio topologico T_0. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. (5) **1**, 84–90 (1940). [MF 2167]

In this note a space is constructed which falls under the classification T_0 of Alexandroff; that is, its system of neighborhoods satisfy the first three postulates of Hausdorff and the postulate of separation of Kolmogoroff but not that of Hausdorff or of Fréchet (so, in particular, is not a space T_2, T_1 or a metric space). The points of the space are locally analytic functions of n-variables and a neighborhood (A, σ) of y_0 is the collection of functions y which are regular at the points of the closed set A, interior to the region of definition of y_0, and at each point of A the values of y and y_0 differ in absolute value by less than $\sigma>0$. An equivalent system (r, σ) of neighborhoods is also defined. *J. F. Randolph.*

Katětov, Miroslav. Über H-abgeschlossene und bikompakte Räume. Časopis Pěst. Mat. Fys. **69**, 36–49 (1940). (German. Czech summary) [MF 1938]

The author obtains results about H-closed spaces, and about compact (bicompact in the sense of Alexandroff and Urysohn) spaces. Considering only H-spaces (Hausdorff spaces), he studies the relations between (1) having a maximal (fewest open sets) Hausdorff topology, (2) being H-closed, (3) having the complements of closures of open sets as a basis. An H-space is compact if and only if (i) distinct points have disjoint closed neighborhoods, and (ii) the topology is maximal. An H-space is compact if and only if each of its closed subsets is H-closed. This result was proved by Stone [Trans. Amer. Math. Soc. **41**, 375–481 (1937)] using the theory of Boolean rings. Every H-space R is a dense subset of an H-closed space S such that each continuous function mapping R into a dense subset Q of an H-space can be extended to some $M\subset S$ in such a way as to cover Q. If Q is compact, we may take $M=S$; S is a topological invariant of R and is called the H-closed hull

of R. Every completely regular space R is a dense subset of a compact space S such that each continuous function mapping R into a dense subset of a compact space Q can be extended to all of S covering all of Q. S is a topological invariant of R and is called the compact hull of R. The paper contains several auxiliary results. *J. W. Tukey.*

Stebakov, S. Über der Fundamentalsatz der Theorie der bikompakten Räume. Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 161–164 (1939). (Russian. German summary) [MF 2111]

The author proves that the classical conditions for bicompactness [Alexandroff and Urysohn, Math. Ann. **92**, 258–266 (1924)] are equivalent in any space in which the whole space and any union of open sets is open. This equivalence was known for Hausdorff spaces.

J. W. Tukey (Princeton, N. J.).

Alexandroff, P. Bikompakte Erweiterungen topologischer Räume. Rec. Math. [Mat. Sbornik] N.S. **5 (47)**, 403–423 (1939). (Russian. German summary) [MF 2309]

The German summary is quite complete, with respect to ideas as well as definitions. The paper is a comprehensive study of the extension of topological spaces in the sense mainly of their bicompactification. The author introduces "regular ends" (completely regular ends) as natural elements in a regular (completely regular) Hausdorff space. A system of open sets is called (completely) regular if for each set of the system there can be found another (which is completely regularly contained in it) whose closure is contained in it. The system is centered if every finite number of its sets has a nonvacuous intersection. A system maximal with respect to both of these properties is a (completely) regular end. Points of the space determine ends: an end not necessarily a point. If R is (completely) regular, the space of its (completely) regular ends is denoted by $(\alpha' R)$ αR. It is proven that $\alpha' R$ coincides with the Čech bicompactification [On bicompact spaces, Ann. of Math. (2) **38**, 823–845 (1937)], denoted by βR: coincidence is to within a homeomorphism leaving R pointwise fixed. It is shown that αR and $\alpha' R$ coincide if and only if αR is bicompact, or is normal, or is completely regular. For normal spaces, αR, $\alpha' R$, βR and also ωR, the Wallman extension [Lattices and topological spaces, Ann. of Math. (2) **39**, 112–127 (1938)] are coincident. The author studies the Wallman extension and establishes that the Čech bicompactification of a completely regular space may be derived from Wallman's in terms of a Hausdorff decomposition [Alexandroff-Hopf, Topologie, Springer, 1935] of ωR, a decomposition which is "most minute" with respect to all Hausdorff decompositions. *L. Zippin* (Flushing, N. Y.).

Hyers, D. H. Locally bounded linear topological spaces. Revista Ci., Lima **41**, 555–574 (1939). [MF 1647]

Continuing his investigation of this subject [cf. Bull. Amer. Math. Soc. **44**, 76–80 (1938)], the author is able to extend Riesz's theory of completely continuous transformations to any linear topological space containing a bounded open set, that is, a locally bounded space. This is accomplished by first showing that in such a space one may introduce an upper semi-continuous "absolute value" which has the properties of a norm if one substitutes the following for the triangle inequality: For every $\eta>0$ there is a $\delta>0$ such that $|x|<\delta$, $|y|<\delta$ implies $|x+y|<\eta$. It is then shown that Riesz's theorems are valid when one has an "absolute value" instead of a norm. In order that the author's postulates for a linear topological space be equivalent to Kolmogoroff's, a further separation postulate is needed [cf. H. Whitney, Duke Math. J. **4**, 495–528 (1938), footnote 22]. *J. V. Wehausen* (New York, N. Y.).

Charpentier, Marie. Sur certaines courbes fermées et leurs bouts premiers. Bull. Sci. Math. **63**, 303–307 (1939). [MF 1625]

Main result: Let C be a continuum separating the plane into two regions which have C as their common boundary. If C contains a prime end (in the sense of Carathéodory) which is identical with C, then C is either indecomposable, or else contains two indecomposable continua P and Q such that $P=\overline{C-Q}$, $Q=\overline{C-P}$. *W. Hurewicz.*

Polak, A. I. On open mappings of locally-connected continua. Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 181–183 (1939). (Russian. German summary) [MF 2113]

Proof of the theorem: If an open mapping f of a locally connected continuum X on a Hausdorff space Y is not one to one, then X contains a proper subcontinuum whose image under f is the entire space Y. *W. Hurewicz.*

Radó, T. and Reichelderfer, P. Cyclic transitivity. Duke Math. J. **6**, 474–485 (1940). [MF 2331]

Noting that the relation of conjugacy as used in the cyclic element theory of the structure of connected and locally connected sets is reflexive and symmetric but not in general transitive, the authors develop extensively a weaker type of transitivity which, in particular, is satisfied by the conjugacy relation. A binary relation R is said to be cyclically transitive provided that, for every finite cyclically ordered set of distinct points $A_1, A_2, \cdots, A_n$ satisfying $A_1RA_2RA_3 \cdots RA_nRA_1$, we have A_iRA_j for every choice of i and j. Two important ways of generating all possible reflexive, symmetric and cyclically transitive binary relations in a space l are given. In terms of an arbitrary reflexive, symmetric and cyclically transitive binary relation R in a space l, concepts such as proper cyclic elements, cut points, end points, H-sets and cyclic chains are introduced. Properties of these are developed showing much similarity to the corresponding notions as used in the cyclic element theory to which they reduce in case the space is topologized and is connected and locally connected. For example, a non-degenerate set S in l is coherent if every pair of points of S are in the relation R and S is complete if it contains every point which is in the R relation to two distinct points of S. A proper cyclic element, then, is a set which is both coherent and complete. Two distinct proper cyclic elements have at most one point in common; and if a and b are distinct points and aRb, there exists one and only one proper cyclic element containing both of them. A striking example is given, however, showing that in the sense defined by the authors, every point of a certain cyclic element may be a cut point and there may be uncountably many distinct cyclic elements. *G. T. Whyburn* (Charlottesville, Va.).

Dushnik, Ben and Miller, E. W. Concerning similarity transformations of linearly ordered sets. Bull. Amer. Math. Soc. **46**, 322–326 (1940). [MF 1841]

Two linearly ordered sets are said to be similar if there exists a 1-1 correspondence which preserves order. The authors raise two questions: (1) Is every infinite ordered

set similar to a proper subset of itself? (2) If every similarity correspondence f of a linearly ordered set to a subset of itself has the property $f(a) \geqq a$ for every a, then is the set well-ordered? (This is the converse of a classical theorem.) For the countable case the authors prove that the answer to both questions is in the affirmative. A subset of the linear continuum is constructed by the Zermelo axiom to show that this conclusion does not extend to uncountable sets in either case. *W. L. Ayres* (Ann Arbor, Mich.).

Eilenberg, Samuel. On a theorem of P. A. Smith concerning fixed points for periodic transformations. Duke Math. J. 6, 428–437 (1940). [MF 2327]

A theorem of the reviewer [Ann. of Math. (2) **35**, 572–578 (1934)] asserts that a periodic transformation Λ of a space X into itself of prime period p must admit at least one fixed point if sufficiently many homotopy groups of X vanish. The present paper contains a proof of the following stronger theorem in which the homotopy conditions are replaced by homology conditions: assume that X is a separable metric space of finite dimension. Let R be a commutative ring with unit element, which does not contain an inverse of p. If every true cycle in X (coef. R) bounds in X, then Λ leaves fixed at least one point. The hypothesis of finite dimensionality and the choice of R as coefficient domain are justified by examples. Consider a particular Λ. The principal lemma asserts that if K is a simple circuit relative to R and if f is a mapping $f(X) \subset K$ which permutes with Λ, then there exists in X a true cycle C^n ($n = \dim X$) such that $f(C^n)$ does not bound in $\bar{K}$. (Here the condition that true cycles in X bound need only be assumed for dimensions less than n.) The lemma is used also to prove the following theorem which may be considered as a generalization of Borsuk's theorem on antipodic transformations of n-spheres [Fund. Math. **20**, 177–190 (1933)]: let p be a prime and R a commutative ring with unit element such that $pR = 0$. Let K be a simple circuit relative to R and Λ an isomorphic transformation of K into itself of period p. Then every continuous transformation $f(\bar{K}) \subset \bar{K}$ with $f\Lambda = \Lambda f$ has the degree 1. *P. A. Smith* (New York, N. Y.).

Whyburn, G. T. A relation between non-alternating and interior transformations. Bull. Amer. Math. Soc. **46**, 320–321 (1940). [MF 1840]

This note proves the following theorem: Let A be a compact continuum, B be an arc or a dendrite with a finite number of end points, and E be the set of end points of B. Any interior transformation $f(A) = B$ can be factored $f = f_2 f_1$, where f_1 is equivalent to f on $f^{-1}(E)$ and 1–1 elsewhere and f_2 is interior and non-alternating. It is shown that the finiteness of the set E may be generalized to the condition that the collection $[f^{-1}(x)]$ of inverses of points of E is semi-closed. *W. L. Ayres* (Ann Arbor, Mich.).

Hall, D. W. and Whyburn, G. T. Arc- and tree-preserving transformations. Trans. Amer. Math. Soc. **48**, 63–71 (1940). [MF 2505]

A single-valued continuous transformation T on the Peano continuum A is said to be arc-preserving if $T(\alpha)$ is an arc for each arc α of A, and tree-preserving if $T(D)$ is a tree for each tree D of A. The present paper seems to complete the study of arc-preserving and tree-preserving transformations insofar as continuity is assumed. It is shown that tree-preserving is equivalent to $T(\alpha)$ a tree for each arc α of A. This gives as a corollary the theorem of R. G. Simond [Duke Math. J. **4**, 575–589 (1938)] that an arc-preserving transformation is tree-preserving. If T is tree-preserving, for each true cyclic element E' of $T(A)$ there exists a unique cyclic element E of A such that $T(E) = E'$, T is a homeomorphism on E and T^{-1} is single valued on $T(E_n)$, where E_n denotes those points of E which are non-cut points of A. As corollaries the authors obtain the results that if $T(A)$ is cyclic, T is equivalent to a monotonic retract of A onto some cyclic element; and if A and $T(A)$ are both cyclic, T is a homeomorphism. This second result was predicted by G. T. Whyburn in an earlier paper [Amer. J. Math. **58**, 305–312 (1936)]. Arc-preserving is equivalent to tree-preserving plus the condition that $T(C)$ is a cyclic chain for each cyclic chain C of A. Finally it is shown that if T is arc-preserving and E is a cyclic element of A, then $T(E)$ is a cyclic element of $T(A)$ or a free arc of cut points of $T(A)$ and, if $T(E)$ is a true cyclic element, T is a homeomorphism on E. *W. L. Ayres* (Ann Arbor, Mich.).

Roberts, J. H. Two-to-one transformations. Duke Math. J. 6, 256–262 (1940). [MF 1557]

A n-to-1 transformation of a space A in a space B is a transformation for which every point of B is the image of exactly n points. O. G. Harrold has shown [Duke Math. J. **5**, 780–793 (1939); these Rev. **1**, 223] that no continuous 2-to-1 transformation can be defined on an arc. The author establishes an analogous result for closed 2-cells. The existence of a 2-to-1 transformation of the 2-cell is led to result in a contradiction by the study of the set K consisting of those points x of the 2-cell for which the "conjugated" point $s(x)$, that is, the point having the same image as x, is a continuous function of x. Some rather complicated properties of the components of the set K are used in the demonstration. Whether a similar result holds for n-cells remains an open problem. *W. Hurewicz.*

Elsholz, L. Zur Theorie der Invarianten, die zur Bestimmung der unteren Grenze der Anzahl der kritischen Punkte einer stetigen Funktion, die auf einer Mannigfaltigkeit bestimmt ist, dienen können. Rec. Math. [Mat. Sbornik] N.S. **5** (**47**), 551–558 (1939). (Russian. German summary) [MF 1346]

The author deals with integer-valued functions $n(A)$ of a closed set A contained in a manifold M. These functions are subjected to the following conditions: (1) If A is a point, $n(A) = \alpha$, where α is a constant integer. (2) If A is isotopic to B, $n(A) = n(B)$. (3) $n(A+B) = n(A) + n(B) + 1 - \alpha$. (4) $n(\overline{U(A, \epsilon)}) = n(A)$ for sufficiently small ϵ. The author shows that for any twice differentiable function f defined on M, all of whose critical values are isolated, the following relation holds

$$n(f \leqq c+\epsilon) \leqq n(f \leqq c-\epsilon) + n(P_c) + 1 - \alpha.$$

P_c denotes here the set of critical points corresponding to the value $f = c$, and ϵ is chosen so small that there is no critical value different from c between $c-\epsilon$ and $c+\epsilon$. This result remains true if conditions (1)–(4) are replaced by somewhat weaker conditions. From the above result it follows easily that

$$k \geqq n(f \leqq x) + 1 - \alpha,$$

where k is the number of critical points belonging to the set $(f \leqq x)$. Furthermore the author classifies the functions $n(A)$ satisfying the conditions stated above and gives various examples of "invariants" of this kind. One of the consequences of his theory is the following statement: A n-mani-

fold can always be obtained from $l \leqq n+1$ n-cells through an identification-process on their boundaries.

W. Hurewicz (Chapel Hill, N. C.).

Elsholz, L. Die Änderung der Bettischen Zahlen der Niveauflächen einer stetigen Funktion, die auf einer Mannigfaltigkeit definiert ist. Rec. Math. [Mat. Sbornik] N.S. **5** (47), 559–564 (1939). (Russian. German summary) [MF 1347]

Let f be a twice differentiable function defined on a manifold M^n. The author studies the change of the Betti numbers mod 2 of the "level sets" $(f=x)$, when x passes through an isolated critical value of f. Combining the classical results of M. Morse concerning the Betti numbers of the sets $(f \leqq x)$ with Pontrjagin's duality theorem, he establishes the following relations for a critical value c corresponding to exactly one nondegenerate critical point of type k:

$$p^k(f=c-\epsilon)=p^k(f=c+\epsilon)-1, \tag{1}$$

and

$$p^{n-k-1}(f=c-\epsilon)=p^{n-k-1}(f=c+\epsilon)-1,$$

or

$$p^{k-1}(f=c-\epsilon)=p^{k-1}(f=c+\epsilon)+1, \tag{2}$$

and

$$p^{n-k}(f=c-\epsilon)=p^{n-k}(f=c+\epsilon)+1,$$

where p^r denotes the r-dimensional Betti number mod 2 and ϵ is chosen so small that there is no critical value different from c between $c-\epsilon$ and $c+\epsilon$. The Betti numbers of dimensions other than k, $k-1$, $n-k$, $n-k-1$ do not change. In the case $n=2k$ there is a further possibility:

$$p^r(f=c-\epsilon)=p^r(f=c+\epsilon), \qquad \text{for } r=0, 1, 2, \cdots, n. \tag{3}$$

W. Hurewicz (Chapel Hill, N. C.).

Chogoshvili, G. S. On level surfaces and domains of smaller values of a function defined on a bounded manifold. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 635–639 (1939). [MF 2051]

A continuation of a previous paper by the same author [ibid. **22**, 293–297 (1939)], this time concerned with the variation of the Betti numbers of the level surfaces and the domains of smaller values at a critical point of a function on the boundary of a manifold. Assumptions, methods and results are similar to those of Morse and van Schaack [Ann. of Math. (2) **35**, 545–571 (1934)]. *H. E. Robbins.*

Morse, Marston. Rank and span in functional topology. Ann. of Math. (2) **41**, 419–454 (1940). [MF 1822]

This is a contribution to the theory of critical limits (relative minima, minimax values, etc.) of a single-valued real function F defined over a metric space M. The difficulties which arise from admitting functions whose critical points and values are not discrete are here met by considering only those characteristic features of the graph of $y=F(x)$ which possess a given degree of coarseness, or which are of "span" greater than $\epsilon>0$. The various numbers which characterize the situation thus depend on ϵ; they are finite if the connectivities of M are finite. The proofs are simplified and unified by a preliminary abstract treatment of the notion of rank. Let G be an abelian group admitting an operator field Δ. Certain elements u of G not including 0 are to have rank, that is, they are to constitute the domain of definition of a real-valued function $\rho(u)$ satisfying certain conditions of which we shall only mention: (1) if u, v, $u+v$ have rank, then $\rho(u+v) \leqq \max(\rho(u), \rho(v))$; (2) if u and v have unequal ranks, $u+v$ has rank. Of importance are subgroups $g(\rho)$ of G whose elements other than 0 all have the same rank ρ and which are maximal in this respect. Essential theorems: the groups $g(\rho)$ corresponding to a given ρ are isomorphic to each other (in a special way); a subgroup g of G all of whose elements other than 0 have rank decomposes into a direct sum of groups $g(\rho)$.

It may be assumed that $0 \leqq F < 1$ over M. Certain conditions of a topological nature (satisfied in all known applications) are imposed on (M, F). In particular the sets F_c defined by $F \leqq c$ are compact. A chain or cycle is definitely on (d-on) F_c if it is on $F_{c-\epsilon}$ for some $\epsilon>0$. Suppose that, over the coefficient field Δ, u is a Vietoris cycle d-mod F_a which is on F_a without being ~ 0 d-mod F_a on F_a; u is called a cap of height $a(u)$ and is said to be linkable if $\beta u \sim 0$ d-on F_a, β being the boundary operator. The l.u.b. of numbers b such that u is not ~ 0 on F_b d-mod F_a is the superior limit $\sigma(u)$ of u; $\sigma(u)$ is to be ∞ if the b's are unbounded. If u is non-linkable, the g.l.b. of numbers c such that $\beta u \sim 0$ d-on F_a mod F_c is the inferior limit $\tau(u)$. The span of u is $a(u)-\tau(u)$ if u is linkable, otherwise $\sigma(u)-a(u)$. Limits are also attached to certain absolute cycles, to βu, for example, if u is a non-linkable cap of height a; in this case the superior limit of βu equals a.

Let N_k^ϵ be a maximal group of non-linkable k-caps (caps of dimension k) with spans greater than $\epsilon>0$, group composition being simply chain-addition. Regarding N as a subgroup of the group of formal k-chains, it is verified that the rank function defined by the cap heights satisfies the rank conditions. Hence the groups N_k^ϵ are isomorphic to each other and therefore have equal dimensions n_k^ϵ. As a result of the conditions on (M, F), the n_k^ϵ are finite. There is at least one group N_{k+1}^ϵ such that the correspondence $u \to \beta u$ defines an isomorphism between N_{k+1}^ϵ and a maximal group βN_{k+1}^ϵ of linkable k-caps of finite spans greater than ϵ. The direct sum $\beta N_{k+1}^\epsilon + N_k^\epsilon$ is isomorphic to a maximal group C_k^ϵ of k-caps of finite spans greater than ϵ and conversely every C_k^ϵ is isomorphic to such a sum. Hence $C_k^\epsilon = \dim C_k^\epsilon = n_k^\epsilon + n_{k+1}^\epsilon$, from which it follows that the c^ϵ's satisfy the Morse inequalities

$$c_k^\epsilon - c_{k-1}^\epsilon + \cdots + (-1)^k c_0^\epsilon \geqq 0, \qquad k=0, 1, \cdots.$$

Let P_k be a maximal group of k-caps with infinite spans. The dimension p_k of P_k equals the kth connectivity number of M. Moreover, $C_k^\epsilon + P_k^\epsilon$ is isomorphic to a maximal group M_k^ϵ of k-caps with spans greater than ϵ so that $\dim M_k^\epsilon = m_k^\epsilon = n_k^\epsilon + n_{k+1}^\epsilon + p_k$. Hence, if the m's and p's are finite, the numbers $\mu_k^\epsilon = m_k^\epsilon - p_k$ satisfy the Morse inequalities. The number m_k^ϵ also equals $\sum_a m_k^\epsilon(a)$, where $m_k^\epsilon(a)$, the kth "ϵ-type number" of a, is the dimension of a maximal group of k-caps of span greater than ϵ and height a. If for each k the cap heights are finite in number, considerations of span may be dropped out of the picture. For in this case the spans of k-caps are bounded away from 0 and hence, if ϵ is small enough, m_k^ϵ equals $m_k = \sum_a m_k(a)$, where $m_k(a)$, the "kth type number of a," is a maximal group of k-caps of height a. Moreover p_k and m_k are then finite and therefore the numbers $\mu_k = m_k - p_k$ satisfy the Morse inequalities.

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